トポロジカル相の分類とClifford代数の拡大問題

* Ken Shiozaki
Email: shiozaki@scphys.kyoto-u.ac.jp

問題設定

与えられた対称性群とガンマ行列（運動項）とハミルトニアン（質量項）からなる群から、Clifford代数の拡大問題としての構造を見つければよい。

このゲームのルールは

- 他の全てと可換な構造はセクターに分離、または複素化
- 全てがお互いに反可換になるように組み替える
- ハミルトニアンは拡大問題の行き先として特別視する。つまりハミルトニアンは生成子の中で一度のみ現れるように組む
- Real classを考える場合は純虚数 J を含める

注意
このノートでは連続系のみ扱い、格子系に特徴的なstaggeredなdisorderに対する不安定性は考えない。

AZ class

Complex class

$$\{e_i, e_j\} = 2\delta_{ij} \qquad Cl_q \to Cl_{q+1} \quad : \quad C_q$$

$$\{H, \Gamma\} = 0$$

class A

H

$$Cl_0 \to Cl_1 \quad : \quad C_0$$

class AIII

H

Γ

$$Cl_1 \to Cl_2 \quad : \quad C_1$$

Real class

$$\{e_i, e_j\} = 0 \quad (i \neq j)$$

$$e_1^2 = \cdots e_p^2 = -1, \quad e_{p+1}^2 = \cdots = e_{p+q}^2 = 1$$

$$J^2 = -1, \quad T^2 = \epsilon_T, \quad C^2 = \epsilon_C$$

$$[H, J] = [H, T] = \{H, C\} = \{J, T\} = \{J, C\} = [T, C] = 0$$

$$Cl_{p,q} \to Cl_{p,q+1} \quad : \quad R_{q-p}$$

$$Cl_{p,q} \to Cl_{p+1,q} \quad : \quad R_{p+2-q}$$

$-$
JH

T ϵ_T JT ϵ_T

class AI : $\epsilon_T = 1, \quad Cl_{0,2} \to Cl_{1,2} \; : \; R_0$

class AII : $\epsilon_T = -1, \quad Cl_{2,0} \to Cl_{3,0} \; : \; R_4$

$+$
H

C ϵ_C JC ϵ_C

class D : $\epsilon_C = 1, \quad Cl_{0,2} \to Cl_{0,3} \; : \; R_2$

class C : $\epsilon_C = -1, \quad Cl_{2,0} \to Cl_{2,1} \; : \; R_6$

$+$
H

ϵ_C C   JCT $-\epsilon_C\epsilon_T$

JC
ϵ_C

class BDI : $\epsilon_T = \epsilon_C = 1, \quad Cl_{1,2} \to Cl_{1,3} \; : \; R_1$

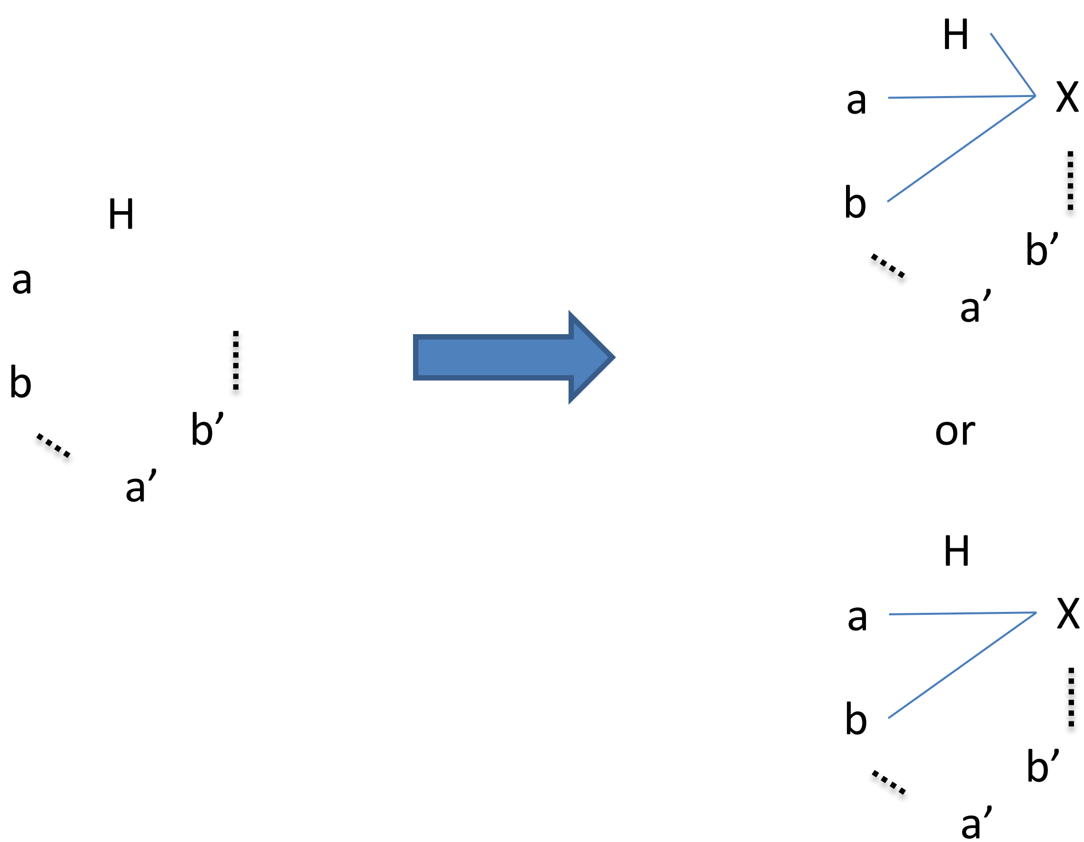
class DIII : $\epsilon_T = -1, \epsilon_C = 1, \quad Cl_{0,3} \to Cl_{0,4} \; : \; R_3$

class CII : $\epsilon_T = \epsilon_C = -1, \quad Cl_{3,0} \to Cl_{3,1} \; : \; R_5$

class CI : $\epsilon_T = 1, \epsilon_C = -1, \quad Cl_{2,1} \to Cl_{2,2} \; : \; R_7$

Additional Symmetry

反可換な図式にひとつ生成子を加える



は可換な関係を表す
空白は反可換な関係を表す

Xを反可換な組に加えたときに

- Xを含めて反可換に組める
- セクターに分離する、もしくは複素化する

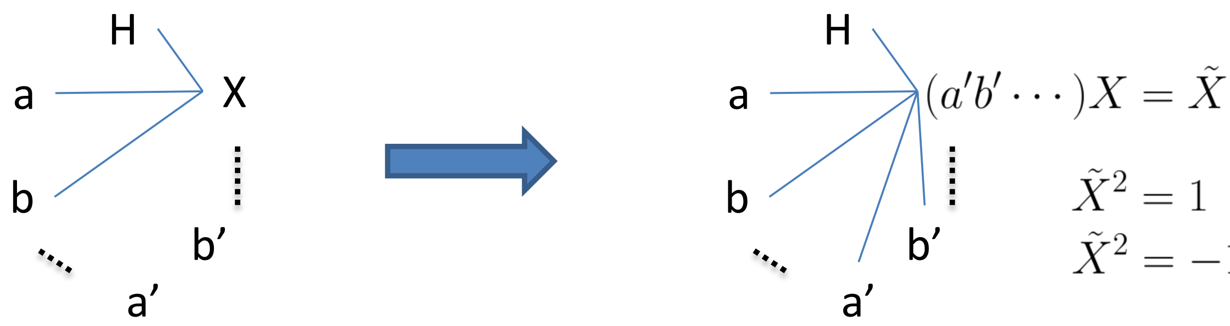
を判定するには

[X,H]=0の場合 → 反可換な線の数
{X,H}=0の場合 → 可換な線の数

を見れば良い

[X,H]=0

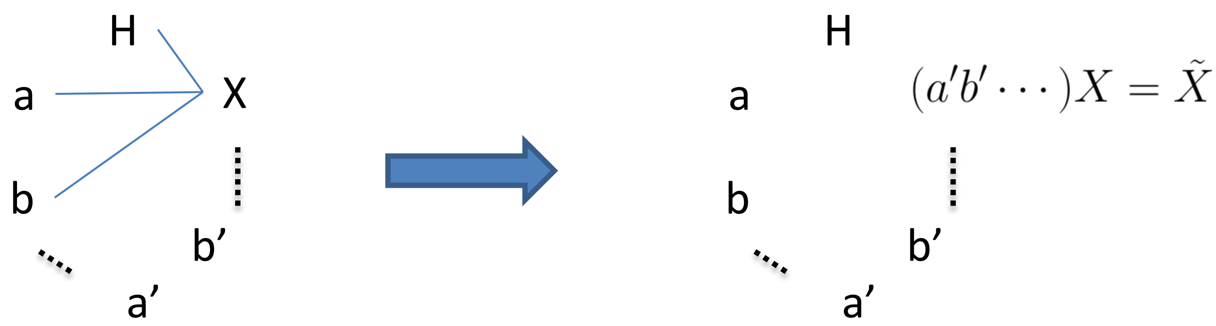
even
a', b', …



$\tilde{X}^2 = 1 \rightarrow decouple$

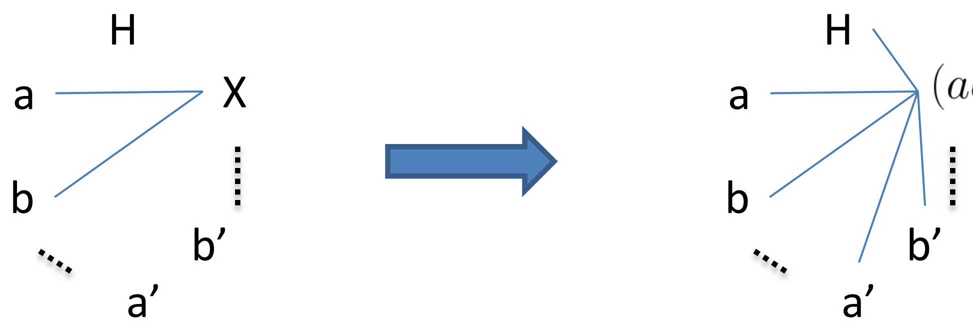
$\tilde{X}^2 = -1 \rightarrow complex$

odd
a', b', …



{X,H}=0

odd
a, b, …

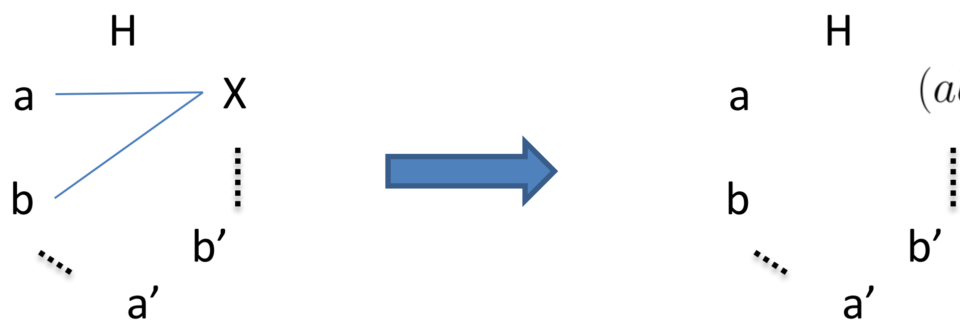


$(ab\cdots)X = \tilde{X}$

$\tilde{X}^2 = 1 \ \rightarrow decouple$

$\tilde{X}^2 = -1 \ \rightarrow complex$

even
a, b, …



$(ab\cdots)X = \tilde{X}$

Application

AZ分類、有限次元

$$H_d = \sum_{i=1}^{d} \gamma_i k_i + H$$

$$\gamma_i^2 = 1$$

$$\{H, \Gamma\} = [H, J] = [H, T] = \{H, C\} = \{J, T\} = \{J, C\} = [T, C] = 0$$

$$\{\gamma_i, H\} = [\gamma_i, J] = \{\gamma_i, T\} = [\gamma_i, C] = \{\gamma_i, \Gamma\} = 0$$

Complex

H

C_d

γ_2

γ_1

H

Γ

C_{d+1}

γ_2

γ_1

Real

JH

T

TJ

γ_2

γ_1

H

C

JC

$J\gamma_2$

$J\gamma_1$

H

C

JC

JCT

$J\gamma_2$

$J\gamma_1$

$$R_q \to R_{q-d}$$

[AZ] Classification table (Schnyder-Ryu-Furusaki-Ludwig, Kitaev)

class	TRS	PHS	chiral	Cq or Rq	d=0	d=1	d=2	d=3	d=4	d=5	d=6	d=7
A				C0	Z		Z		Z		Z	
AIII			1	C1		Z		Z		Z		Z
AI	1			R0	Z				Z		Z2	Z2
BDI	1	1		R1	Z2	Z				Z		Z2
D		1		R2	Z2	Z2	Z				Z	
DIII	-1	1		R3		Z2	Z2	Z				Z
AII	-1			R4	Z		Z2	Z2	Z			
CII	-1	-1		R5		Z		Z2	Z2	Z		
C		-1		R6			Z		Z2	Z2	Z	
CI	1	-1		R7				Z		Z2	Z2	Z

Defect zero mode (Teo-Kane 2010)

D : co-dimension of defect

$$H_{d,D} = \sum_{i=1}^{d} \gamma_i k_i + \sum_{a=1}^{D} \tilde{\gamma}_a s_a + H$$

$$\gamma_i^2 = \tilde{\gamma}_a^2 = 1$$

$$\{H, \Gamma\} = [H, J] = [H, T] = \{H, C\} = \{J, T\} = \{J, C\} = [T, C] = 0$$

$$\{\gamma_i, H\} = [\gamma_i, J] = \{\gamma_i, T\} = [\gamma_i, C] = \{\gamma_i, \Gamma\} = 0$$

$$\{\tilde{\gamma}_a, H\} = [\tilde{\gamma}_a, J] = [\tilde{\gamma}_a, T] = \{\tilde{\gamma}_a, C\} = \{\tilde{\gamma}_a, \Gamma\} = 0$$

or

$$\{J\tilde{\gamma}_a, H\} = [J\tilde{\gamma}_a, J] = \{J\tilde{\gamma}_a, T\} = [J\tilde{\gamma}_a, C] = \{J\tilde{\gamma}_a, \Gamma\} = 0$$

Complex

H ⋯

$\tilde{\gamma}_2$

γ_1 $\tilde{\gamma}_1$

γ_2 ⋯

C_{d+D}

H ⋯

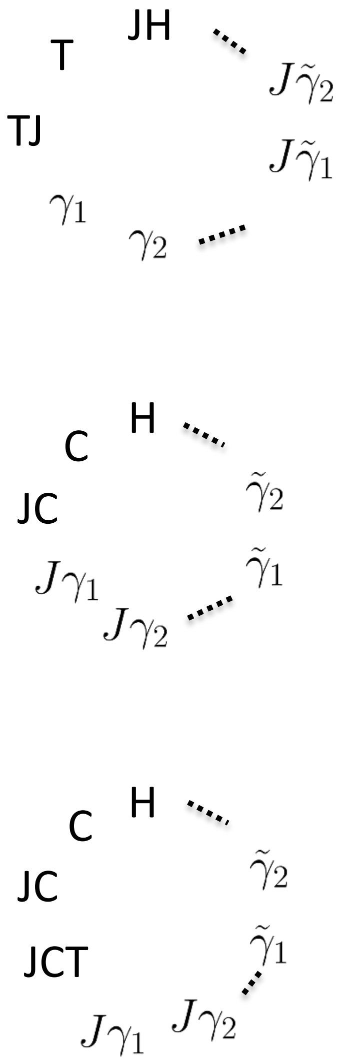
Γ $\tilde{\gamma}_2$

γ_1 $\tilde{\gamma}_1$

γ_2 ⋯

C_{d+D+1}

Real



$$R_q \to R_{q-d+D}$$

[TK] Classification table for defect zero mode (Teo-Kane)

$$\delta = d - D$$

D : co-dimension of defect

class	TRS	PHS	chiral	Cq or Rq	δ=0	δ=1	δ=2	δ=3	δ=4	δ=5	δ=6	δ=7
A				C0	Z		Z		Z		Z	
AIII			1	C1		Z		Z		Z		Z
AI	1			R0	Z				Z		Z2	Z2
BDI	1	1		R1	Z2	Z				Z		Z2
D		1		R2	Z2	Z2	Z				Z	
DIII	-1	1		R3		Z2	Z2	Z				Z
AII	-1			R4	Z		Z2	Z2	Z			
CII	-1	-1		R5		Z		Z2	Z2	Z		
C		-1		R6			Z		Z2	Z2	Z	
CI	1	-1		R7				Z		Z2	Z2	Z

Single additional symmetry

Reflection type (波数を奇数個反転)

$$H_d = \sum_{l=1}^{2n-1} \gamma_l k_l + \sum_{i=2n}^{d} \gamma_j k_j + H$$

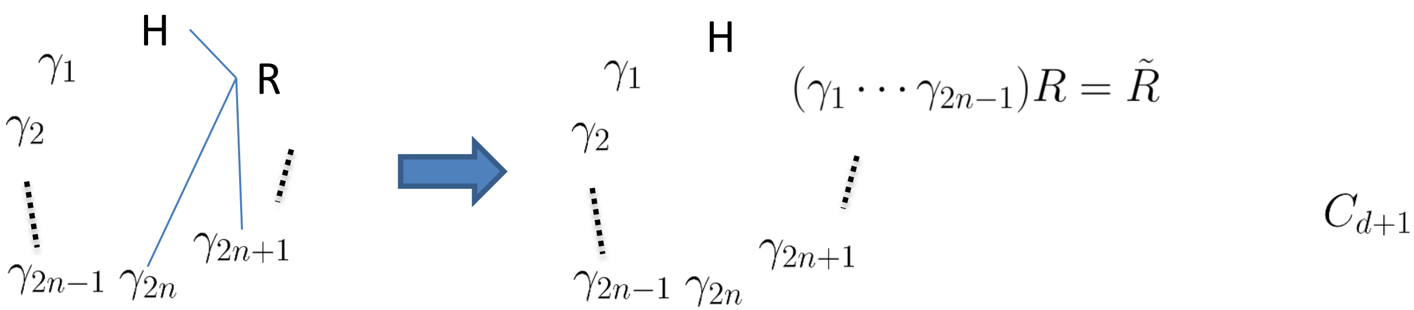
$$\gamma_i^2 = 1$$

$$\{H, \Gamma\} = [H, J] = [H, T] = \{H, C\} = \{J, T\} = \{J, C\} = [T, C] = 0$$

$$\{\gamma_i, H\} = [\gamma_i, J] = \{\gamma_i, T\} = [\gamma_i, C] = \{\gamma_i, \Gamma\} = 0$$

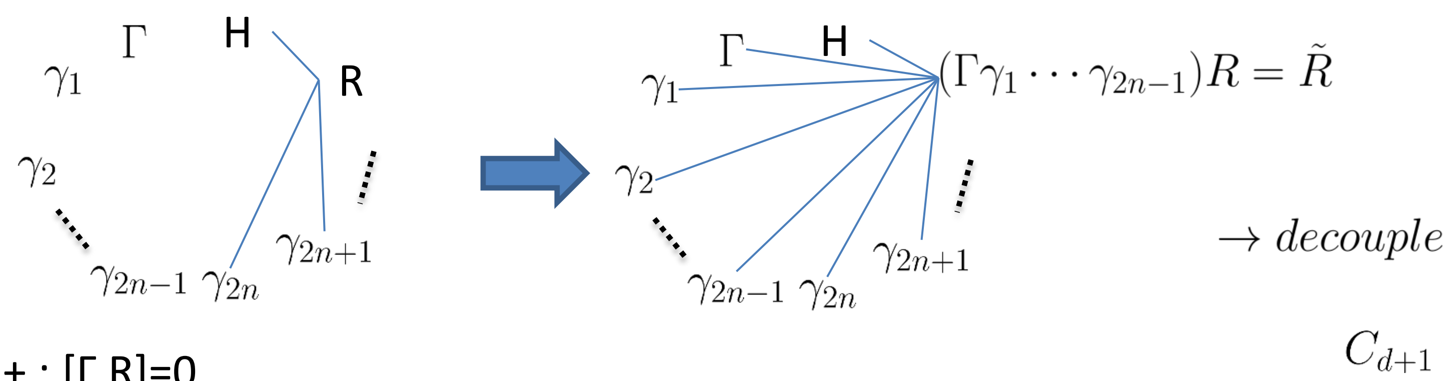
$$[R, H] = [R, J] = \{R, \gamma_l\} = [R, \gamma_j] = 0$$

Complex, class A

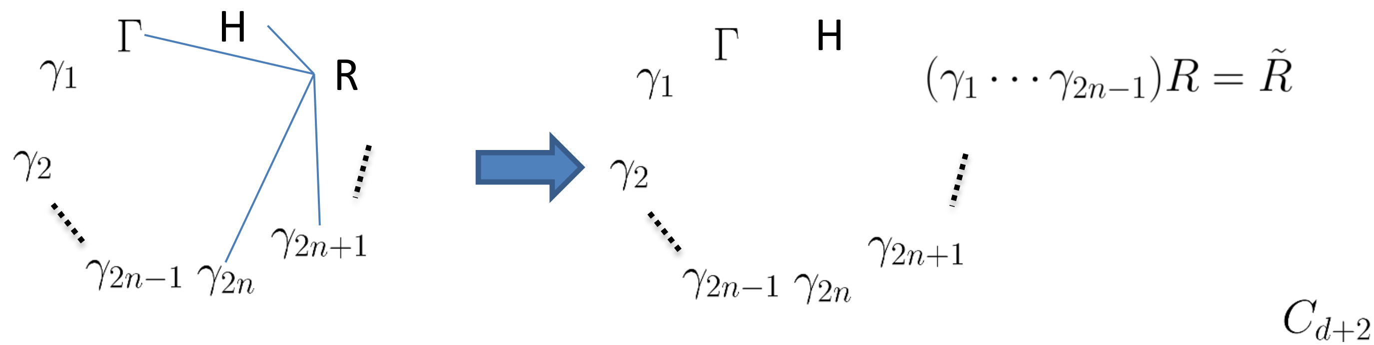


Complex, class AIII

R- : {Γ,R}=0

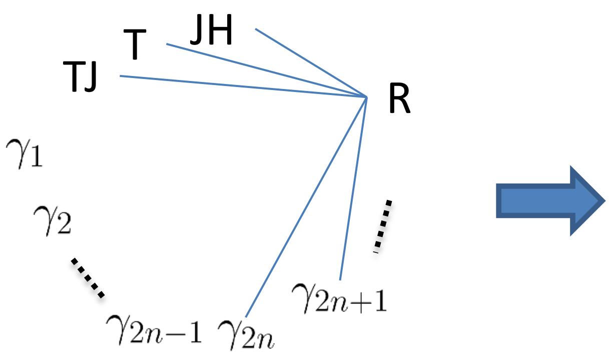


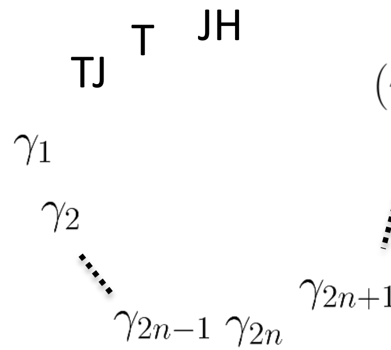
R+ : [Γ,R]=0



Real,   class AI, AII

R+ : [T,R]=0

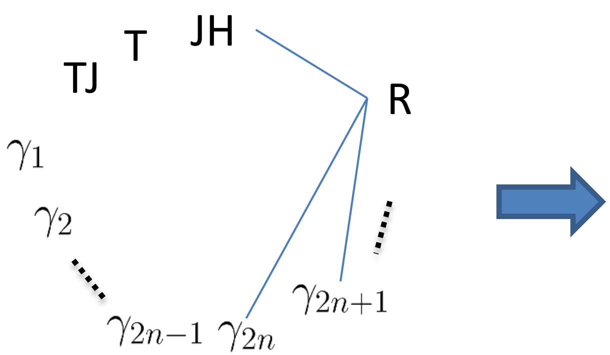


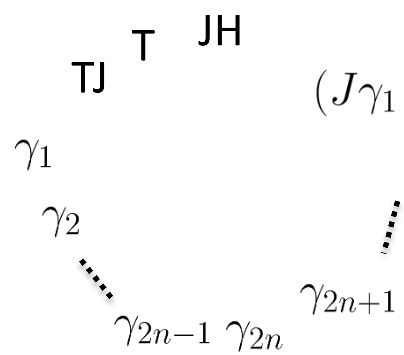


$$(\gamma_1 \cdots \gamma_{2n-1})R = \tilde{R}$$

$$\tilde{R}^2 = (-1)^n R^2$$

R- : {T,R}=0



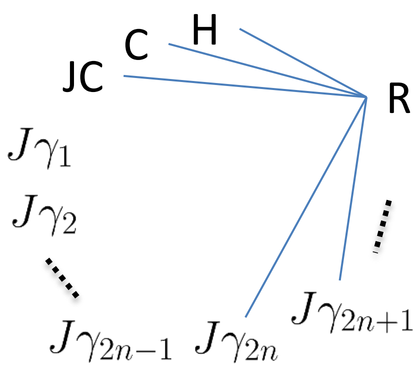


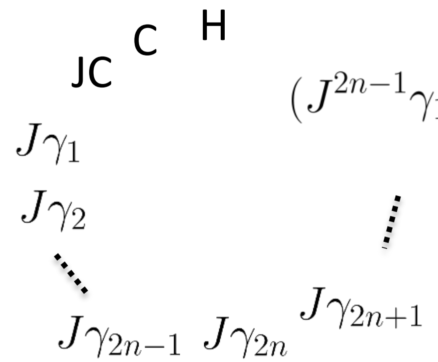
$$(J\gamma_1 \cdots \gamma_{2n-1})R = \tilde{R}$$

$$\tilde{R}^2 = -(-1)^n R^2$$

Real, class D, C

R+ : [C,R]=0

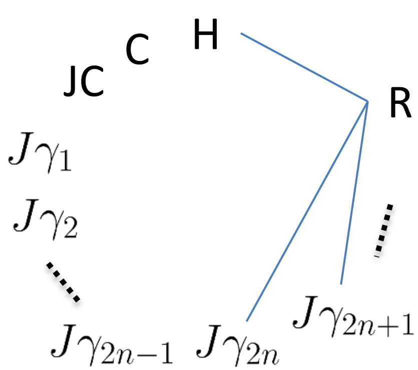


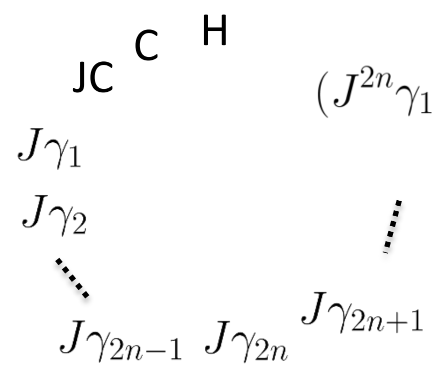


$$(J^{2n-1}\gamma_1 \cdots \gamma_{2n-1})R = \tilde{R}$$

$$\tilde{R}^2 = -(-1)^n R^2$$

R- : {C,R}=0



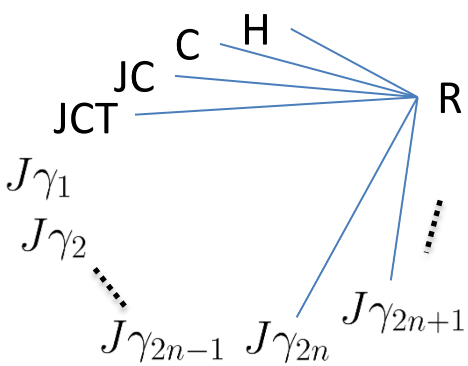


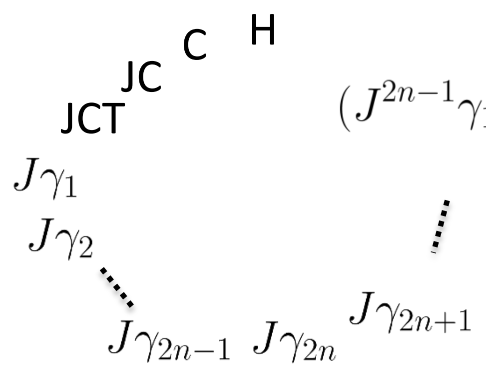
$$(J^{2n}\gamma_1 \cdots \gamma_{2n-1})R = \tilde{R}$$

$$\tilde{R}^2 = (-1)^n R^2$$

Real, class BDI, DIII, CII, CI

R++ : [T,R]=[C,R]=0

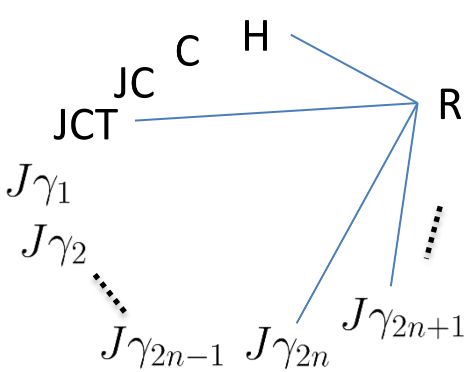


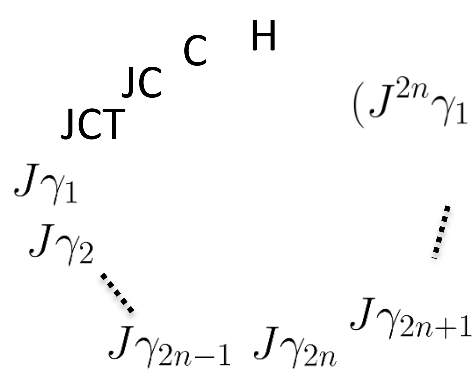


$$(J^{2n-1}\gamma_1 \cdots \gamma_{2n-1})R = \tilde{R}$$

$$\tilde{R}^2 = -(-1)^n R^2$$

R-- : {T,R}={C,R}=0



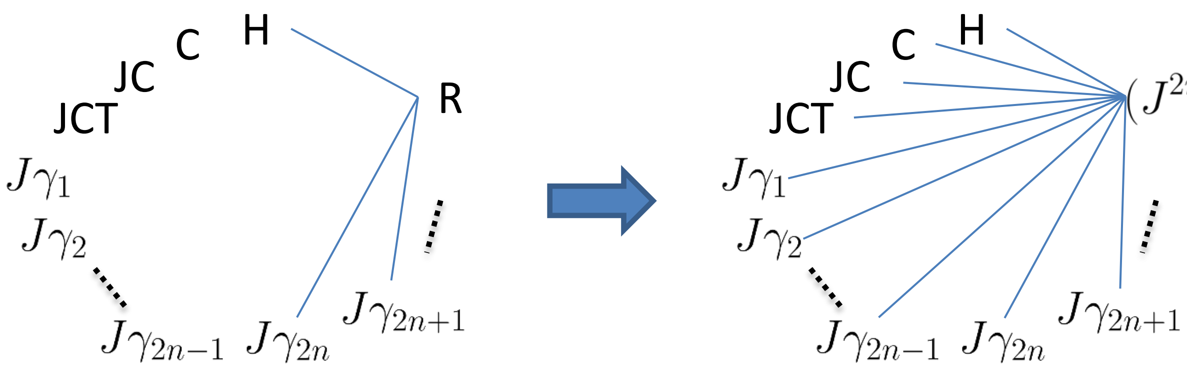


$$(J^{2n}\gamma_1 \cdots \gamma_{2n-1})R = \tilde{R}$$

$$\tilde{R}^2 = (-1)^n R^2$$

Real, class BDI, DIII, CII, CI

R+- : [T,R]={C,R}=0

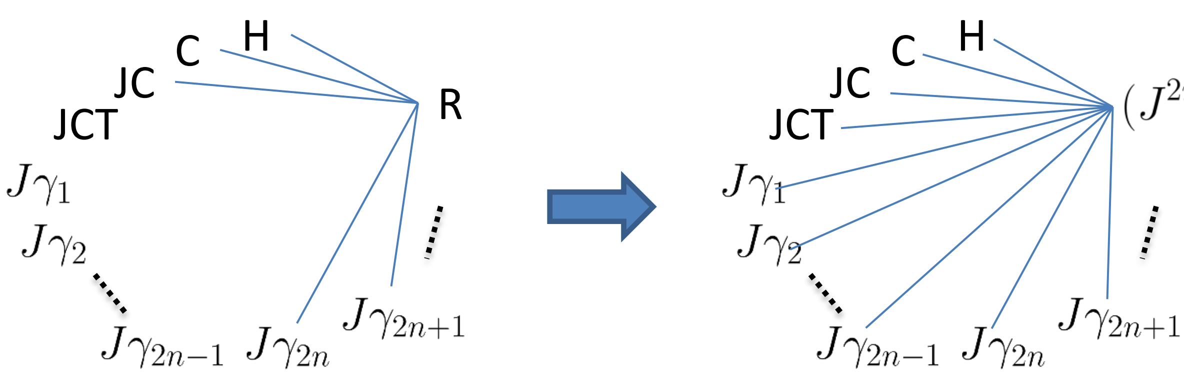


$$(J^{2n-1}CT\gamma_1\cdots\gamma_{2n-1})R = \tilde{R}$$

$$\tilde{R}^2 = -(-1)^n \epsilon_T \epsilon_C R^2$$

$$\tilde{R}^2 = 1 \;\rightarrow decouple$$
$$\tilde{R}^2 = -1 \;\rightarrow complex$$

R-+ : {T,R}=[C,R]=0



$$(J^{2n}CT\gamma_1\cdots\gamma_{2n-1})R = \tilde{R}$$

$$\tilde{R}^2 = (-1)^n \epsilon_T \epsilon_C R^2$$

$$\tilde{R}^2 = 1 \;\rightarrow decouple$$
$$\tilde{R}^2 = -1 \;\rightarrow complex$$

[R1] Classification table for $(-1)^n R^2 = 1$ $R_{\eta_\Gamma},\ R_{\eta_T},\ R_{\eta_C},\ R_{\eta_T\eta_C}$ 反転する波数が2n-1 個

Reflection		class	Cq or Rq	d=0	d=1	d=2	d=3	d=4	d=5	d=6	d=7
R	+1	A	C1		Z		Z		Z		Z
R+	+1	AIII	C0	Z		Z		Z		Z	
R-	decouple	AIII	C1		Z		Z		Z		Z
R+, R++	-1	AI	R7				Z		Z2	Z2	Z
		BDI	R0	Z				Z		Z2	Z2
		D	R1	Z2	Z				Z		Z2
		DIII	R2	Z2	Z2	Z				Z	
		AII	R3		Z2	Z2	Z				Z
		CII	R4	Z		Z2	Z2	Z			
		C	R5		Z		Z2	Z2	Z		
		CI	R6			Z		Z2	Z2	Z	
R-, R--	+1	AI	R1	Z2	Z				Z		Z2
		BDI	R2	Z2	Z2	Z				Z	
		D	R3		Z2	Z2	Z				Z
		DIII	R4	Z		Z2	Z2	Z			
		AII	R5		Z		Z2	Z2	Z		
		CII	R6			Z		Z2	Z2	Z	
		C	R7				Z		Z2	Z2	Z
		CI	R0	Z				Z		Z2	Z2
R-+	decouple	BDI	R1	Z2	Z				Z		Z2
R+-		DIII	R3		Z2	Z2	Z				Z
R-+		CII	R5		Z		Z2	Z2	Z		
R+-		CI	R7				Z		Z2	Z2	Z
R-+	complex	DIII, CI	C1		Z		Z		Z		Z
R+-		BDI, CII	C1		Z		Z		Z		Z

[R2] Classification table for $(-1)^n R^2 = -1$ $R_{\eta_\Gamma},\ R_{\eta_T},\ R_{\eta_C},\ R_{\eta_T \eta_C}$ 反転する波数が2n-1 個 (Chiu-Yao-Ryu)

Reflection		class	Cq or Rq	d=0	d=1	d=2	d=3	d=4	d=5	d=6	d=7
R	+1	A	C1		Z		Z		Z		Z
R+	+1	AIII	C0	Z		Z		Z		Z	
R-	decouple	AIII	C1		Z		Z		Z		Z
R+, R++	+1	AI	R1	Z2	Z				Z		Z2
		BDI	R2	Z2	Z2	Z				Z	
		D	R3		Z2	Z2	Z				Z
		DIII	R4	Z		Z2	Z2	Z			
		AII	R5		Z		Z2	Z2	Z		
		CII	R6			Z		Z2	Z2	Z	
		C	R7				Z		Z2	Z2	Z
		CI	R0	Z				Z		Z2	Z2
R-, R--	-1	AI	R7				Z		Z2	Z2	Z
		BDI	R0	Z				Z		Z2	Z2
		D	R1	Z2	Z				Z		Z2
		DIII	R2	Z2	Z2	Z				Z	
		AII	R3		Z2	Z2	Z				Z
		CII	R4	Z		Z2	Z2	Z			
		C	R5		Z		Z2	Z2	Z		
		CI	R6			Z		Z2	Z2	Z	
R+-	decouple	BDI	R1	Z2	Z				Z		Z2
R-+		DIII	R3		Z2	Z2	Z				Z
R+-		CII	R5		Z		Z2	Z2	Z		
R-+		CI	R7				Z		Z2	Z2	Z
R-+	complex	BDI, CII	C1		Z		Z		Z		Z
R+-		DIII, CI	C1		Z		Z		Z		Z

Π-rotation type (波数を偶数(ゼロを含む)個反転)

$$H_d = \sum_{l=1}^{2n} \gamma_l k_l + \sum_{i=2n+1}^{d} \gamma_j k_j + H$$

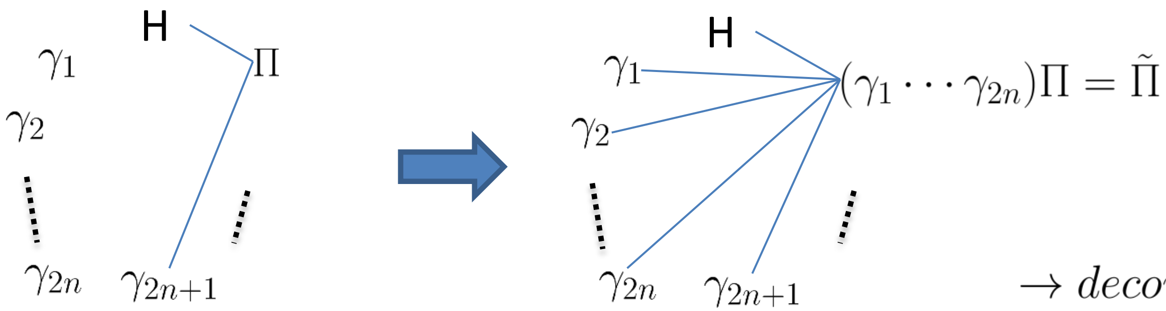
$$\gamma_i^2 = 1$$

$$\{H, \Gamma\} = [H, J] = [H, T] = \{H, C\} = \{J, T\} = \{J, C\} = [T, C] = 0$$

$$\{\gamma_i, H\} = [\gamma_i, J] = \{\gamma_i, T\} = [\gamma_i, C] = \{\gamma_i, \Gamma\} = 0$$

$$[\Pi, H] = [\Pi, J] = \{\Pi, \gamma_l\} = [\Pi, \gamma_j] = 0$$

Complex,   class A

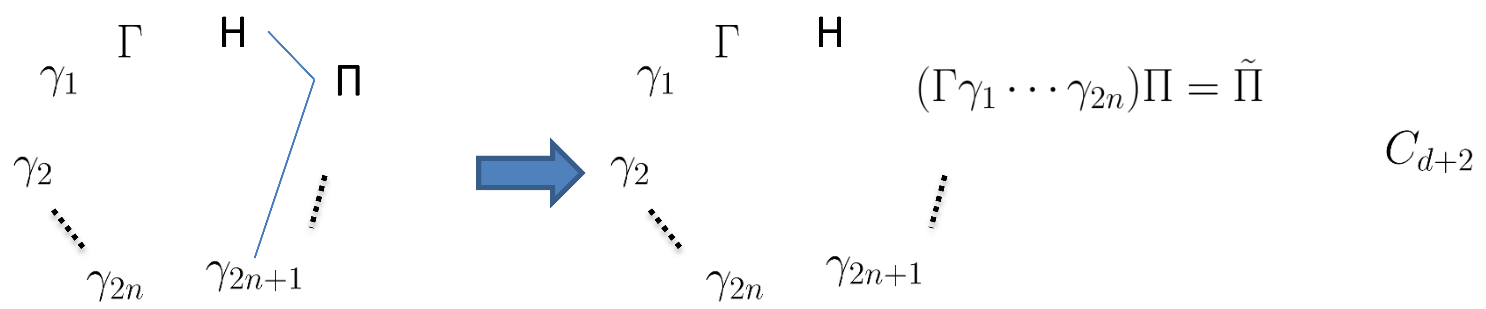


$\to decouple$

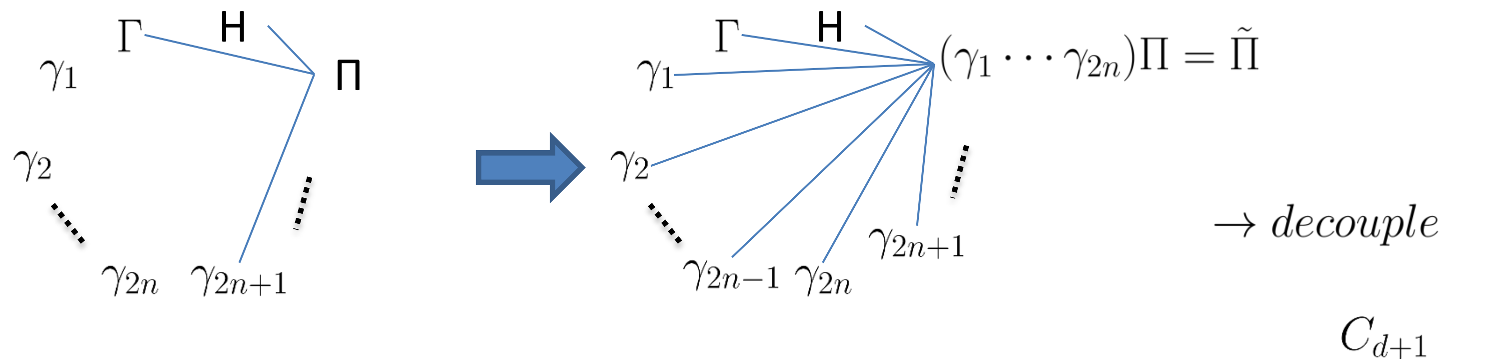
C_d

Complex, class AIII

Π- : {Γ,Π}=0

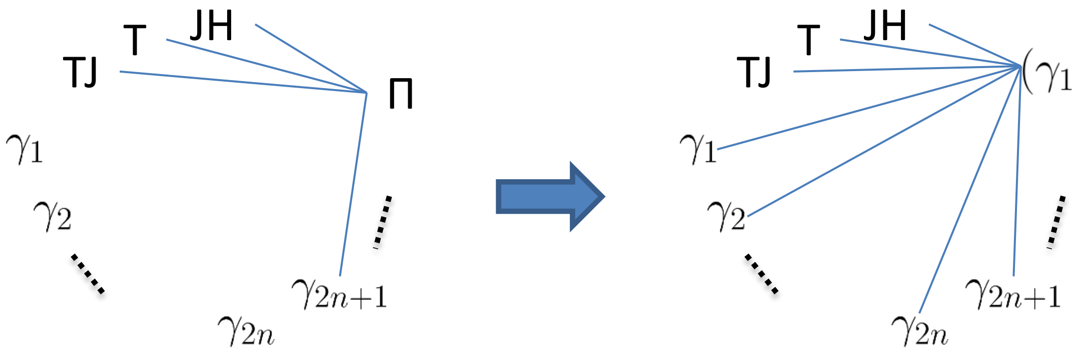


Π+ : [Γ,Π]=0



Real, class AI, AII

Π+ : [T,Π]=0

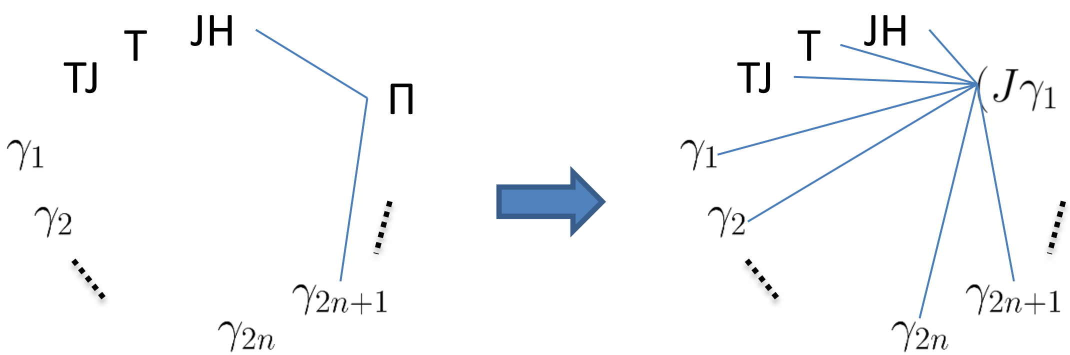


$$(\gamma_1 \cdots \gamma_{2n})\Pi = \tilde{\Pi}$$

$$\tilde{\Pi}^2 = (-1)^n \Pi^2$$

$$\tilde{\Pi}^2 = 1 \quad \to decouple$$
$$\tilde{\Pi}^2 = -1 \quad \to complex$$

Π- : {T,Π}=0



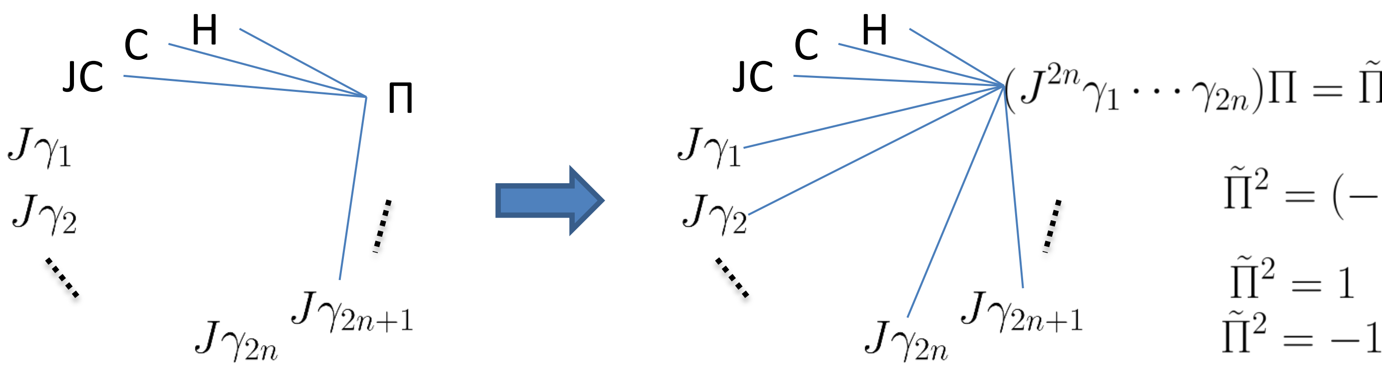
$$(J\gamma_1 \cdots \gamma_{2n})\Pi = \tilde{\Pi}$$

$$\tilde{\Pi}^2 = -(-1)^n \Pi^2$$

$$\tilde{\Pi}^2 = 1 \quad \to decouple$$
$$\tilde{\Pi}^2 = -1 \quad \to complex$$

Real, class D, C

Π+ : [C,Π]=0

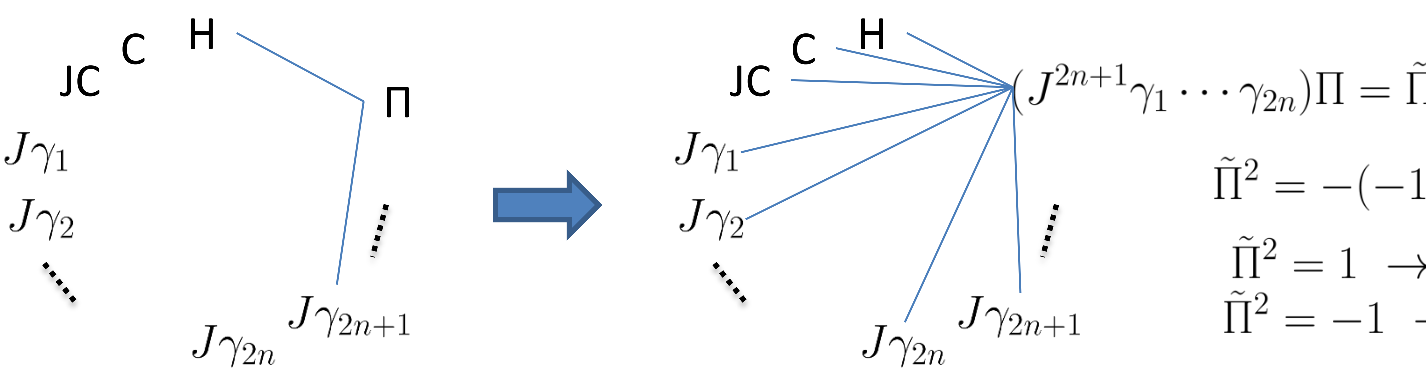


$$\tilde{\Pi}^2 = (-1)^n \Pi^2$$

$$\tilde{\Pi}^2 = 1 \quad \to decouple$$

$$\tilde{\Pi}^2 = -1 \quad \to complex$$

Π- : {C,Π}=0



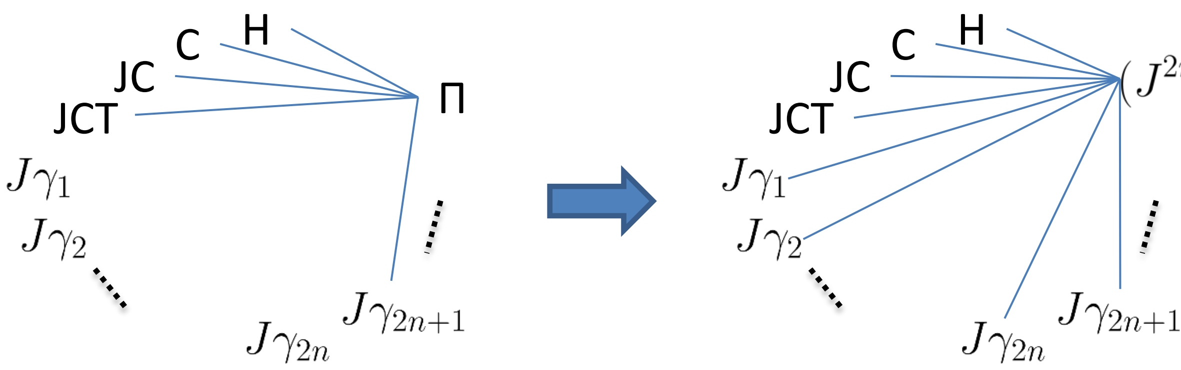
$$\tilde{\Pi}^2 = -(-1)^n \Pi^2$$

$$\tilde{\Pi}^2 = 1 \quad \to decouple$$

$$\tilde{\Pi}^2 = -1 \quad \to complex$$

Real, class BDI, DIII, CII, CI

Π++ : [T,Π]=[C,Π]=0

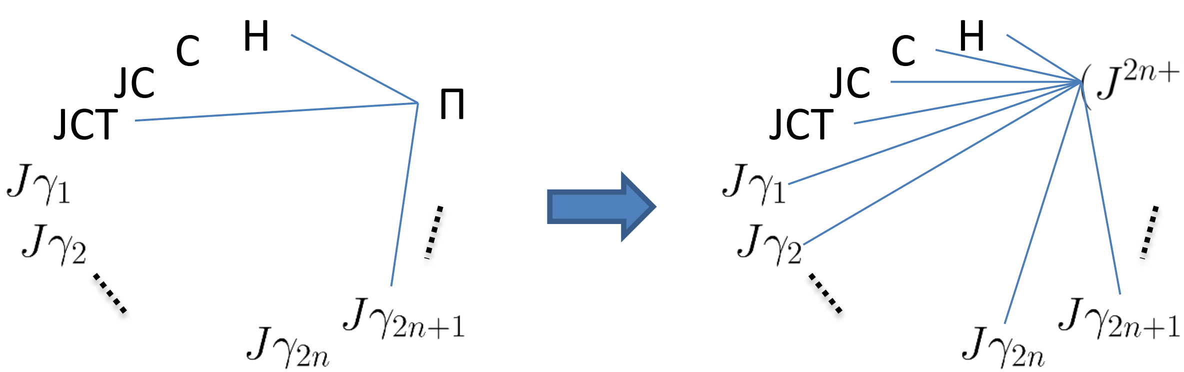


$$\tilde{\Pi}^2 = (-1)^n \Pi^2$$

$$\tilde{\Pi}^2 = 1 \ \rightarrow decouple$$

$$\tilde{\Pi}^2 = -1 \ \rightarrow complex$$

Π-- : {T,Π}={C,Π}=0



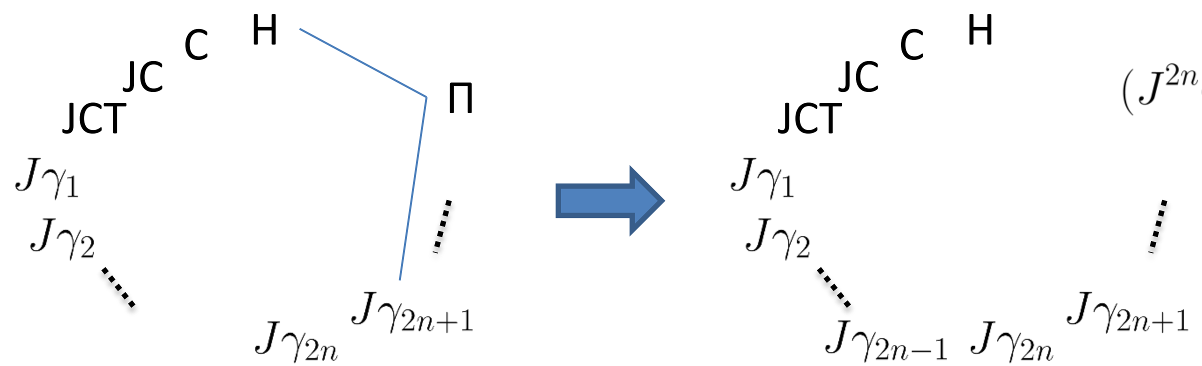
$$\tilde{\Pi}^2 = -(-1)^n \Pi^2$$

$$\tilde{\Pi}^2 = 1 \ \rightarrow decouple$$

$$\tilde{\Pi}^2 = -1 \ \rightarrow complex$$

Real, class BDI, DIII, CII, CI

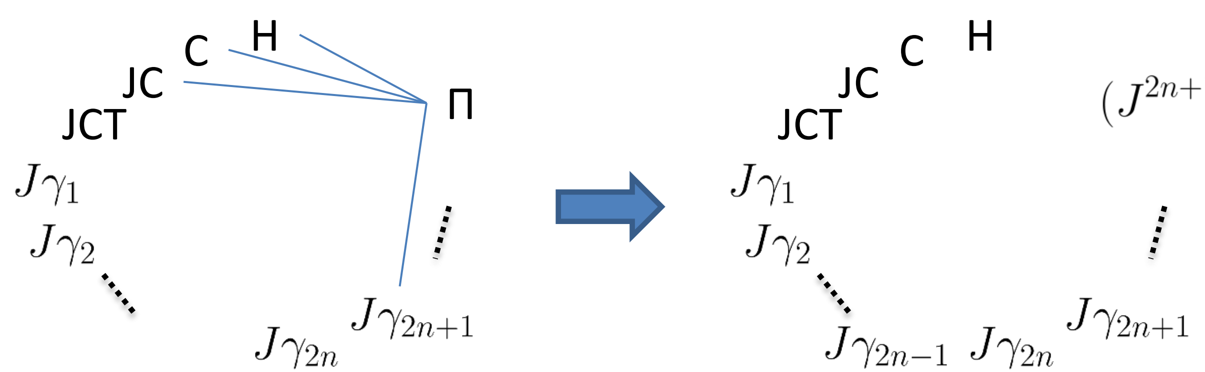
Π+- : [T,Π]={C,Π}=0



$$(J^{2n}CT\gamma_1\cdots\gamma_{2n})\Pi = \tilde{\Pi}$$

$$\tilde{\Pi}^2 = -(-1)^n \epsilon_T \epsilon_C \Pi^2$$

Π-+ : {T,Π}=[C,Π]=0



$$(J^{2n+1}CT\gamma_1\cdots\gamma_{2n})\Pi = \tilde{\Pi}$$

$$\tilde{\Pi}^2 = (-1)^n \epsilon_T \epsilon_C \Pi^2$$

[Π1] Classification table for $(-1)^n\Pi^2 = 1$ $\Pi_{\eta_\Gamma}, \Pi_{\eta_T}, \Pi_{\eta_C}, \Pi_{\eta_T\eta_C}$
反転する波数が 2n 個

Reflection		class	Cq or Rq	d=0	d=1	d=2	d=3	d=4	d=5	d=6	d=7
Π	decouple	A	C0	Z		Z		Z		Z	
Π+	decouple	AIII	C1		Z		Z		Z		Z
Π-	+1	AIII	C0	Z		Z		Z		Z	
Π+, Π++	decouple	AI	R0	Z				Z		Z2	Z2
		BDI	R1	Z2	Z				Z		Z2
		D	R2	Z2	Z2	Z				Z	
		DIII	R3		Z2	Z2	Z				Z
		AII	R4	Z		Z2	Z2	Z			
		CII	R5		Z		Z2	Z2	Z		
		C	R6			Z		Z2	Z2	Z	
		CI	R7				Z		Z2	Z2	Z
Π-, Π--	complex	AI, D, AII, C	C0	Z		Z		Z		Z	
		BDI, DIII, CII, CI	C1		Z		Z		Z		Z
Π+-	-1	BDI	R0				Z		Z2	Z2	Z
	+1	DIII	R4		Z2	Z2	Z				Z
	+1	CII	R6		Z		Z2	Z2	Z		
	-1	CI	R6		Z		Z2	Z2	Z		
Π-+	+1	BDI	R2	Z2	Z				Z		Z2
	-1	DIII	R2	Z2	Z				Z		Z2
	-1	CII	R4	Z		Z2	Z2	Z			
	+1	CI	R0				Z		Z2	Z2	Z

[Π2] Classification table for $(-1)^n\Pi^2=-1$ $\Pi_{\eta_\Gamma},\ \Pi_{\eta_T},\ \Pi_{\eta_C},\ \Pi_{\eta_T\eta_C}$
反転する波数が 2n 個

Reflection		class	Cq or Rq	d=0	d=1	d=2	d=3	d=4	d=5	d=6	d=7
Π	decouple	A	C0	Z		Z		Z		Z	
Π+	decouple	AIII	C1		Z		Z		Z		Z
Π-	+1	AIII	C0	Z		Z		Z		Z	
Π-, Π--	decouple	AI	R0	Z				Z		Z2	Z2
		BDI	R1	Z2	Z				Z		Z2
		D	R2	Z2	Z2	Z				Z	
		DIII	R3		Z2	Z2	Z				Z
		AII	R4	Z		Z2	Z2	Z			
		CII	R5		Z		Z2	Z2	Z		
		C	R6			Z		Z2	Z2	Z	
		CI	R7				Z		Z2	Z2	Z
Π+, Π++	complex	AI, D, AII, C	C0	Z		Z		Z		Z	
		BDI, DIII, CII, CI	C1		Z		Z		Z		Z
Π-+	-1	BDI	R0	Z				Z		Z2	Z2
	+1	DIII	R4	Z		Z2	Z2	Z			
	+1	CII	R6			Z		Z2	Z2	Z	
	-1	CI	R6			Z		Z2	Z2	Z	
Π+-	+1	BDI	R2	Z2	Z				Z		Z2
	-1	DIII	R2	Z2	Z				Z		Z2
	-1	CII	R4		Z2	Z2	Z				Z
	+1	CI	R0				Z		Z2	Z2	Z

（おまけ）

Inversion type (全ての波数を反転) （→reflectionとΠ-rotationの表の組み換え）

$$H_d = \sum_{i=1}^{d} \gamma_i k_i + H$$

$$\gamma_i^2 = 1$$

$$\{H,\Gamma\} = [H,J] = [H,T] = \{H,C\} = \{J,T\} = \{J,C\} = [T,C] = 0$$

$$\{\gamma_i, H\} = [\gamma_i, J] = \{\gamma_i, T\} = [\gamma_i, C] = \{\gamma_i, \Gamma\} = 0$$

$$[I,H] = [I,J] = \{I,\gamma_j\} = 0$$

[I1] Classification table for $I^2=1$ I_{η_Γ}, I_{η_T}, I_{η_C}, $I_{\eta_T\eta_C}$ 全ての波数を反転

class	Inversion	d=0		d=1		d=2		d=3		d=4		d=5		d=6		d=7	
A	I	decouple	Z	+1	Z	decouple	Z	+1	Z	decouple	Z	+1	Z	decouple	Z	+1	Z
AIII	I+	decouple		+1		decouple		+1		decouple	Z	+1		decouple		+1	
	I-	+1	Z	decouple	Z	+1	Z	decouple	Z	+1	Z	decouple	Z	+1	Z	decouple	Z
AI	I+	decouple	Z	+1	Z	complex	Z	-1	Z	decouple	Z	+1	Z	complex	Z	-1	Z
	I-	complex	Z	-1		decouple		+1		complex	Z	-1	Z2	decouple	Z2	+1	Z2
AII	I+	decouple	Z	+1	Z	complex	Z	-1	Z	decouple	Z	+1	Z	complex	Z	-1	Z
	I-	complex	Z	-1	Z2	decouple	Z2	+1	Z2	complex	Z	-1		decouple		+1	
D	I+	decouple	Z2	+1	Z2	complex	Z	-1		decouple		+1		complex	Z	-1	Z2
	I-	complex	Z	-1	Z	decouple	Z	+1	Z	complex	Z	-1	Z	decouple	Z	+1	Z
C	I+	decouple		+1		complex	Z	-1	Z2	decouple	Z2	+1	Z2	complex	Z	-1	
	I-	complex	Z	-1	Z	decouple	Z	+1	Z	complex	Z	-1	Z	decouple	Z	+1	Z
BDI	I++	decouple	Z2	+1	Z2	complex		-1		decouple		+1		complex		-1	Z2
	I--	complex		-1		decouple		+1		complex		-1		decouple		+1	
	I+-	-1		decouple	Z	+1		complex	Z	-1		decouple	Z	+1		complex	Z
	I-+	+1	Z2	complex	Z	-1		decouple		+1		complex	Z	-1	Z2	decouple	Z2
DIII	I++	decouple		+1		complex		-1		decouple		+1		complex		-1	
	I--	complex		-1	Z2	decouple	Z2	+1	Z2	complex		-1		decouple		+1	
	I+-	+1		complex	Z	-1		decouple	Z	+1		complex	Z	-1		decouple	Z
	I-+	-1	Z2	decouple	Z2	+1	Z2	complex	Z	-1		decouple		+1		complex	Z
CI	I++	decouple		+1		complex		-1		decouple		+1		complex		-1	
	I--	complex		-1		decouple		+1		complex		-1	Z2	decouple	Z2	+1	Z2
	I+-	-1		complex	Z	+1		decouple	Z	-1	Z2	complex	Z	+1	Z2	decouple	Z
	I-+	+1		decouple		-1		complex	Z	+1		decouple	Z2	-1		complex	Z
CII	I++	decouple		+1		complex		-1	Z2	decouple	Z2	+1	Z2	complex		-1	
	I--	complex		-1		decouple		+1		complex		-1		decouple		+1	
	I+-	+1		decouple	Z	-1	Z2	complex	Z	+1	Z2	decouple	Z	-1		complex	Z
	I-+	-1	Z	complex	Z	+1		decouple	Z2	-1		complex	Z	+1		decouple	

[I2] Classification table for $I^2=-1$ I_{η_Γ}, I_{η_T}, I_{η_C}, $I_{\eta_T\eta_C}$ 全ての波数を反転

class	Inversion	d=0		d=1		d=2		d=3		d=4		d=5		d=6		d=7	
A	I	decouple	Z	+1	Z	decouple	Z	+1	Z	decouple	Z	+1	Z	decouple	Z	+1	Z
AIII	I+	decouple		+1		decouple		+1		decouple	Z	+1		decouple		+1	
	I-	+1	Z	decouple	Z	+1	Z	decouple	Z	+1	Z	decouple	Z	+1	Z	decouple	Z
AI	I-	decouple	Z	+1	Z	complex	Z	-1	Z	decouple	Z	+1	Z	complex	Z	-1	Z
	I+	complex	Z	-1		decouple		+1		complex	Z	-1	Z2	decouple	Z2	+1	Z2
AII	I-	decouple	Z	+1	Z	complex	Z	-1	Z	decouple	Z	+1	Z	complex	Z	-1	Z
	I+	complex	Z	-1	Z2	decouple	Z2	+1	Z2	complex	Z	-1		decouple		+1	
D	I-	decouple	Z2	+1	Z2	complex	Z	-1		decouple		+1		complex	Z	-1	Z2
	I+	complex	Z	-1	Z	decouple	Z	+1	Z	complex	Z	-1	Z	decouple	Z	+1	Z
C	I-	decouple		+1		complex	Z	-1	Z2	decouple	Z2	+1	Z2	complex	Z	-1	
	I+	complex	Z	-1	Z	decouple	Z	+1	Z	complex	Z	-1	Z	decouple	Z	+1	Z
BDI	I--	decouple	Z2	+1	Z2	complex		-1		decouple		+1		complex		-1	Z2
	I++	complex		-1		decouple		+1		complex		-1		decouple		+1	
	I-+	-1		decouple	Z	+1		complex	Z	-1		decouple	Z	+1		complex	Z
	I+-	+1	Z2	complex	Z	-1		decouple		+1		complex	Z	-1	Z2	decouple	Z2
DIII	I--	decouple		+1		complex		-1		decouple		+1		complex		-1	
	I+	complex		-1	Z2	decouple	Z2	+1	Z2	complex		-1		decouple		+1	
	I-+	+1		complex	Z	-1		decouple	Z	+1		complex	Z	-1		decouple	Z
	I+-	-1	Z2	decouple	Z2	+1	Z2	complex	Z	-1		decouple		+1		complex	Z
CI	I--	decouple		+1		complex		-1		decouple		+1		complex		-1	
	I++	complex		-1		decouple		+1		complex		-1	Z2	decouple	Z2	+1	Z2
	I-+	-1		complex	Z	+1		decouple	Z	-1	Z2	complex	Z	+1	Z2	decouple	Z
	I+-	+1		decouple		-1		complex	Z	+1		decouple	Z2	-1		complex	Z
CII	I--	decouple		+1		complex		-1	Z2	decouple	Z2	+1	Z2	complex		-1	
	I++	complex		-1		decouple		+1		complex		-1		decouple		+1	
	I-+	+1		decouple	Z	-1	Z2	complex	Z	+1	Z2	decouple	Z	-1		complex	Z
	I+-	-1	Z	complex	Z	+1		decouple	Z2	-1		complex	Z	+1		decouple	

Anti-linear Reflection type (反転しない波数が奇数個)

$$H_d = \sum_{l=1}^{2n-1} \gamma_l k_l + \sum_{i=2n}^{d} \gamma_j k_j + H$$

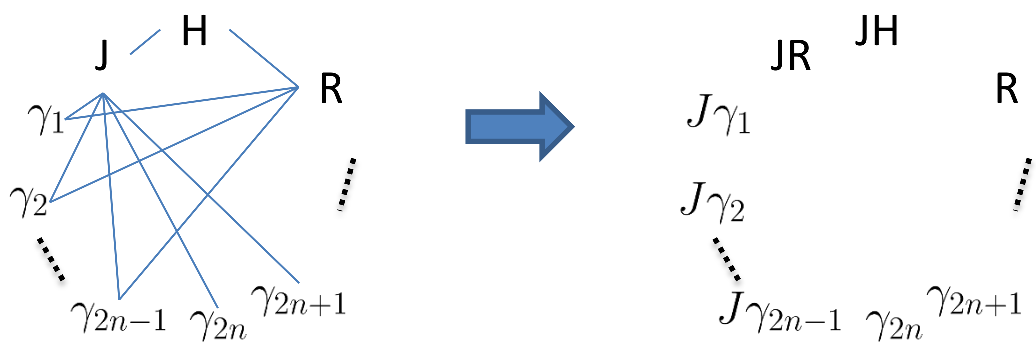
$$\gamma_i^2 = 1$$

$$\{H, \Gamma\} = [H, J] = [H, T] = \{H, C\} = \{J, T\} = \{J, C\} = [T, C] = 0$$

$$\{\gamma_i, H\} = [\gamma_i, J] = \{\gamma_i, T\} = [\gamma_i, C] = \{\gamma_i, \Gamma\} = 0$$

$$[R, H] = \underline{\{R, J\}} = \underline{[R, \gamma_l]} = \underline{\{R, \gamma_j\}} = 0$$

Complex, class A →半線形な対称性によりRealに

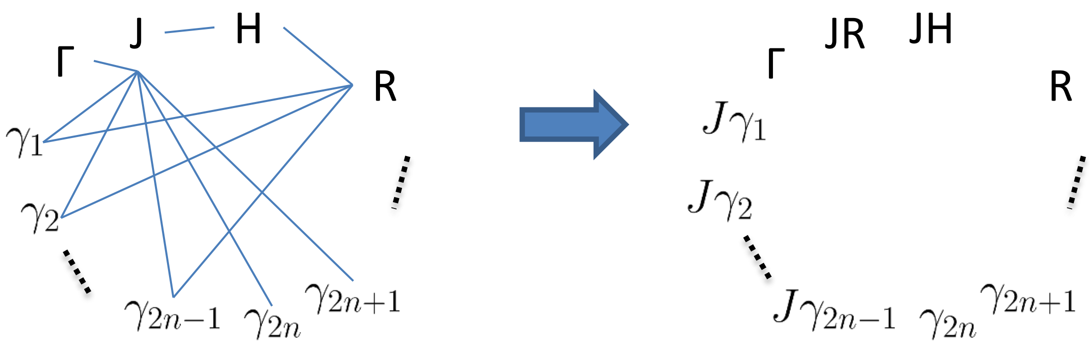


$$R^2 = 1 \; : \; Cl_{2n-1,d-2n+3} \to Cl_{2n,d-2n+3} \qquad R_{4n-d-2}$$

$$R^2 = -1 \; : \; Cl_{2n+1,d-2n+1} \to Cl_{2n+2,d-2n+1} \qquad R_{4n-d+2}$$

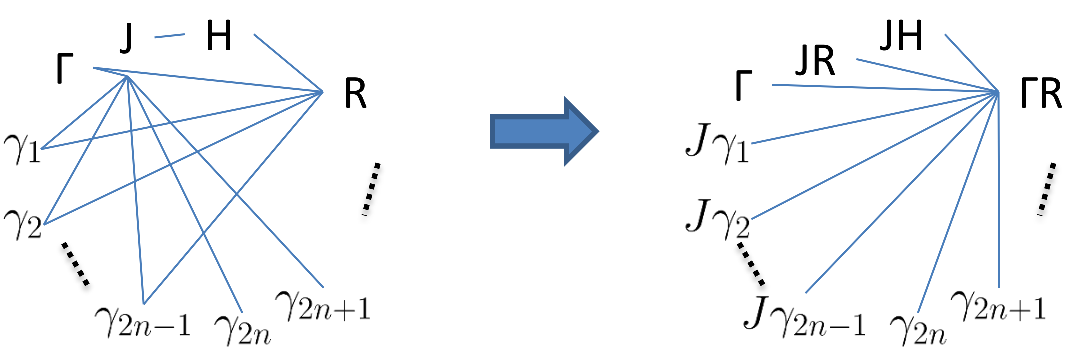
Complex, class AIII →半線形な対称性によりRealに

R- : {Γ,R}=0



$$R^2 = 1 \ : \ Cl_{2n-1,d-2n+4} \rightarrow Cl_{2n,d-2n+4} \qquad R_{4n-d-3}$$
$$R^2 = -1 \ : \ Cl_{2n+1,d-2n+2} \rightarrow Cl_{2n+2,d-2n+2} \qquad R_{4n-d+1}$$

R+ : [Γ,R]=0

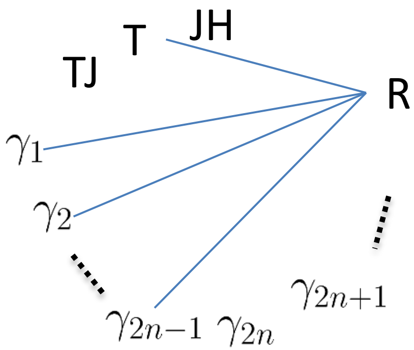


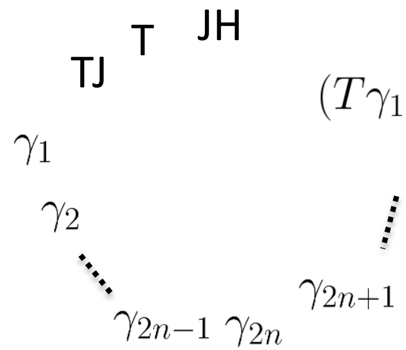
間違い

$$R^2 = 1 \ : decouple, \ Cl_{2n-1,d-2n+3} \rightarrow Cl_{2n,d-2n+3} \qquad R_{4n-d-2}$$
$$R^2 = -1 \ : complex, \ Cl_{d+2} \rightarrow Cl_{d+3} \qquad C_{d+2}$$

Real, class AI, AII

R+ : [T,R]=0

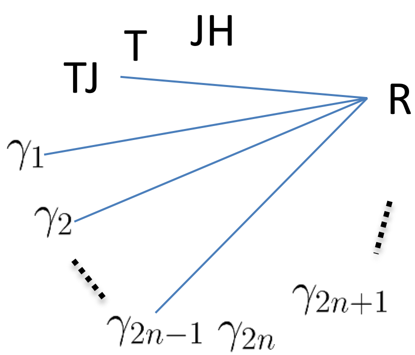


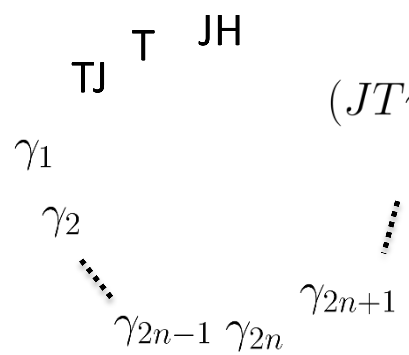


$$(T\gamma_1 \cdots \gamma_{2n-1})R = \tilde{R}$$

$$\tilde{R}^2 = \epsilon_T(-1)^n R^2$$

R- : {T,R}=0



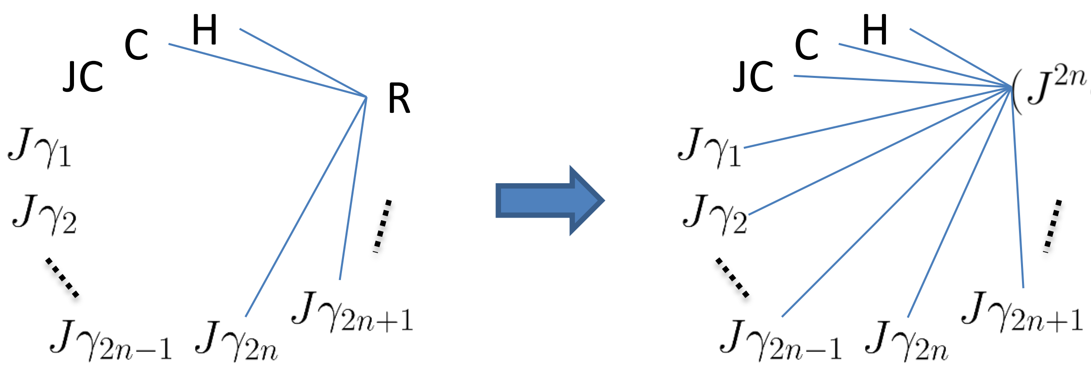


$$(JT\gamma_1 \cdots \gamma_{2n-1})R = \tilde{R}$$

$$\tilde{R}^2 = -\epsilon_T(-1)^n R^2$$

Real, class D, C

R+ : [C,R]=0

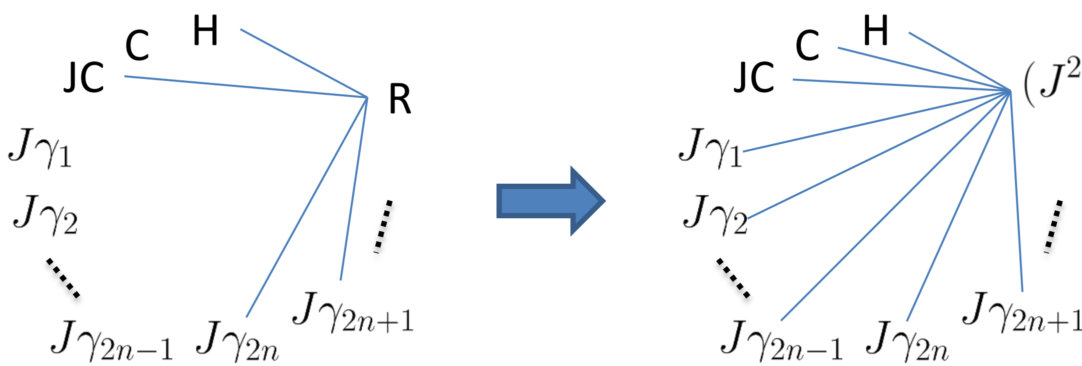


$$(J^{2n}C\gamma_1\cdots\gamma_{2n-1})R = \tilde{R}$$

$$\tilde{R}^2 = -\epsilon_C(-1)^n R^2$$

$$\tilde{R}^2 = 1 \quad \rightarrow decouple$$
$$\tilde{R}^2 = -1 \quad \rightarrow complex$$

R- : {C,R}=0



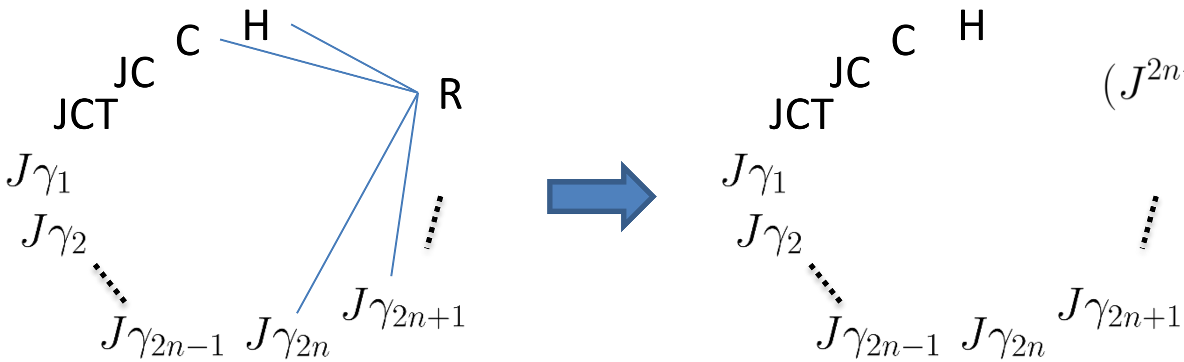
$$(J^{2n-1}C\gamma_1\cdots\gamma_{2n-1})R = \tilde{R}$$

$$\tilde{R}^2 = -\epsilon_C(-1)^n R^2$$

$$\tilde{R}^2 = 1 \quad \rightarrow decouple$$
$$\tilde{R}^2 = -1 \quad \rightarrow complex$$

Real,   class BDI, DIII, CII, CI

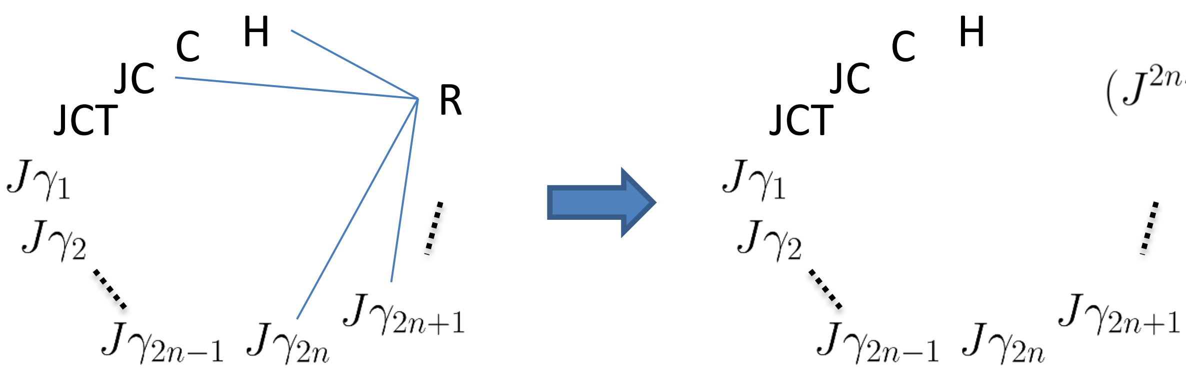
R++ : [T,R]=[C,R]=0



$$(J^{2n-1}T\gamma_1\cdots\gamma_{2n-1})R = \tilde{R}$$

$$\tilde{R}^2 = -\epsilon_T(-1)^n R^2$$

R-- : {T,R}={C,R}=0

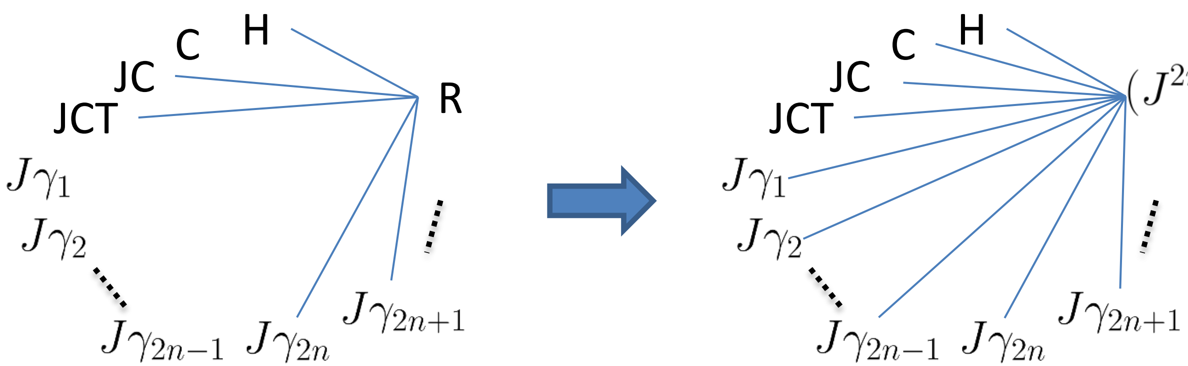


$$(J^{2n}T\gamma_1\cdots\gamma_{2n-1})R = \tilde{R}$$

$$\tilde{R}^2 = -\epsilon_T(-1)^n R^2$$

Real,　class BDI, DIII, CII, CI

R+- : [T,R]={C,R}=0

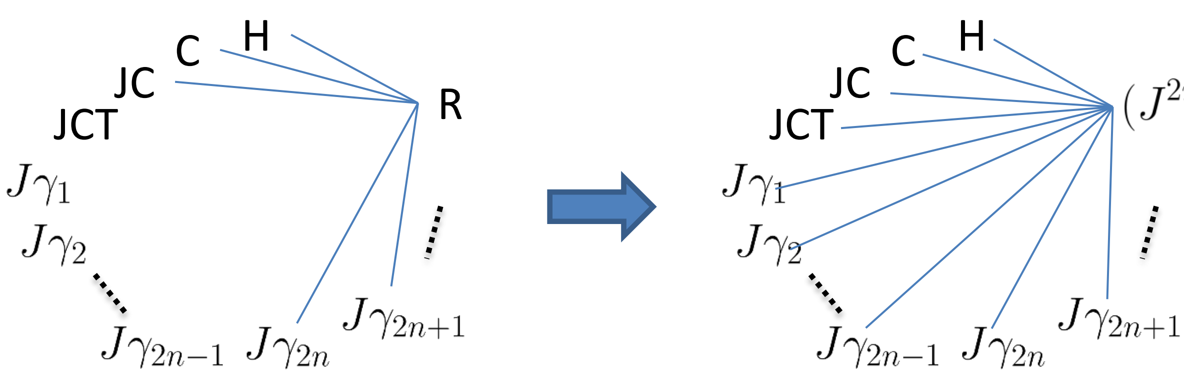


$$(J^{2n-1}C\gamma_1\cdots\gamma_{2n-1})R=\tilde{R}$$

$$\tilde{R}^2=-\epsilon_C(-1)^n R^2$$

$$\tilde{R}^2=1\ \rightarrow decouple$$
$$\tilde{R}^2=-1\ \rightarrow complex$$

R-+ : {T,R}=[C,R]=0
間違い



$$(J^{2n}CT\gamma_1\cdots\gamma_{2n-1})R=\tilde{R}$$

$$\tilde{R}^2=-(-1)^n\epsilon_T\epsilon_C R^2$$

$$\tilde{R}^2=1\ \rightarrow decouple$$
$$\tilde{R}^2=-1\ \rightarrow complex$$

[aR1] Classification table for anti-linear R $(-1)^n R^2 = 1$ $R_{\eta_\Gamma},\ R_{\eta_T},\ R_{\eta_C},\ R_{\eta_T\eta_C}$
反転しない波数が 2n-1 個

class	Reflection		Cq or Rq	d=0	d=1	d=2	d=3	d=4	d=5	d=6	d=7
A	R	real	R6			Z		Z2	Z2	Z	
AIII	R+, R^2=1	decouple, real	R6			Z		Z2	Z2	Z	
	R+, R^2=-1	complex	C0	Z		Z		Z		Z	
	R-	real	R5		Z		Z2	Z2	Z		
AI	R+	-1	R7				Z		Z2	Z2	Z
	R-	+1	R1	Z2	Z				Z		Z2
BDI	R++, R--	-1	R0	Z				Z		Z2	Z2
	R+-, R-+	complex	C1		Z		Z		Z		Z
D	R	complex	C0	Z		Z		Z		Z	
DIII	R++, R--	+1	R4	Z		Z2	Z2	Z			
	R+-	complex	C1		Z		Z		Z		Z
	R-+	decouple	R3		Z2	Z2	Z				Z
AII	R+	+1	R5		Z		Z2	Z2	Z		
	R-	-1	R3		Z2	Z2	Z				Z
CII	R++, R--	+1	R6			Z		Z2	Z2	Z	
	R+-	decouple	R5		Z		Z2	Z2	Z		
	R-+	complex	C1		Z		Z		Z		Z
C	R	decouple	R6			Z		Z2	Z2	Z	
CI	R++, R--	-1	R6			Z		Z2	Z2	Z	
	R+-, R-+	decouple	R7				Z		Z2	Z2	Z

[aR2] Classification table for anti-linear R $(-1)^n R^2 = -1$ $R_{\eta_\Gamma},\ R_{\eta_T},\ R_{\eta_C},\ R_{\eta_T\eta_C}$
反転しない波数が 2n-1 個

間違い

class	Reflection		Cq or Rq	d=0	d=1	d=2	d=3	d=4	d=5	d=6	d=7
A	R	real	R2	Z2	Z2	Z				Z	
AIII	R+, R^2=1	decouple, real	R2	Z2	Z2	Z				Z	
	R+, R^2=-1	complex	C0	Z		Z		Z		Z	
	R-	real	R1	Z2	Z				Z		Z2
AI	R+	+1	R1	Z2	Z				Z		Z2
	R-	-1	R7				Z		Z2	Z2	Z
BDI	R++, R--	+1	R2	Z2	Z2	Z				Z	
	R+-, R-+	decouple	R1	Z2	Z				Z		Z2
D	R	decouple	R2	Z2	Z2	Z				Z	
DIII	R++, R--	-1	R2	Z2	Z2	Z				Z	
	R+-	decouple	R3		Z2	Z2	Z				Z
	R-+	complex	C1		Z		Z		Z		Z
AII	R+	-1	R3		Z		Z2	Z2	Z		
	R-	+1	R5		Z		Z2	Z2	Z		
CII	R++, R--	-1	R4	Z		Z2	Z2	Z			
	R+-	complex	C1		Z		Z		Z		Z
	R-+	decouple	R5		Z		Z2	Z2	Z		
C	R	complex	C0	Z		Z		Z		Z	
CI	R++, R--	+1	R0			Z		Z2	Z2	Z	
	R+-, R-+	complex	C1		Z		Z		Z		Z

Anti-linear Π-rotation type (反転しない波数が偶数個)

$$H_d = \sum_{l=1}^{2n} \gamma_l k_l + \sum_{i=2n+1}^{d} \gamma_j k_j + H$$

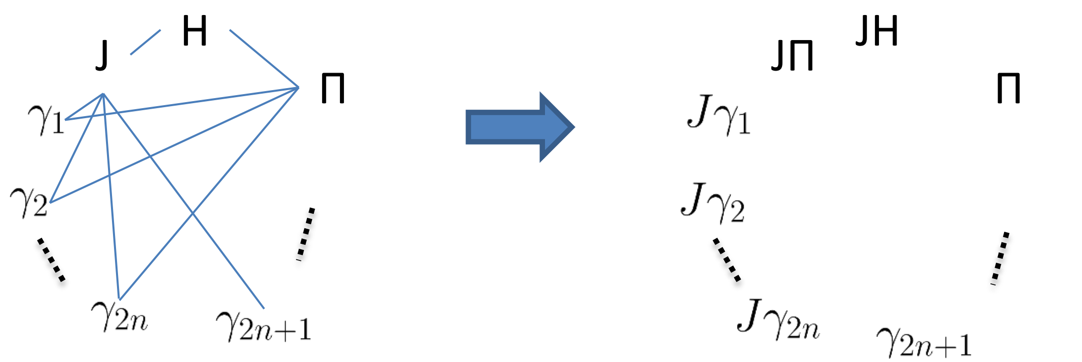
$$\gamma_i^2 = 1$$

$$\{H, \Gamma\} = [H, J] = [H, T] = \{H, C\} = \{J, T\} = \{J, C\} = [T, C] = 0$$

$$\{\gamma_i, H\} = [\gamma_i, J] = \{\gamma_i, T\} = [\gamma_i, C] = \{\gamma_i, \Gamma\} = 0$$

$$[\Pi, H] = \underline{\{\Pi, J\}} = \underline{[\Pi, \gamma_l]} = \underline{\{\Pi, \gamma_j\}} = 0$$

Complex, class A →半線形な対称性によりRealに

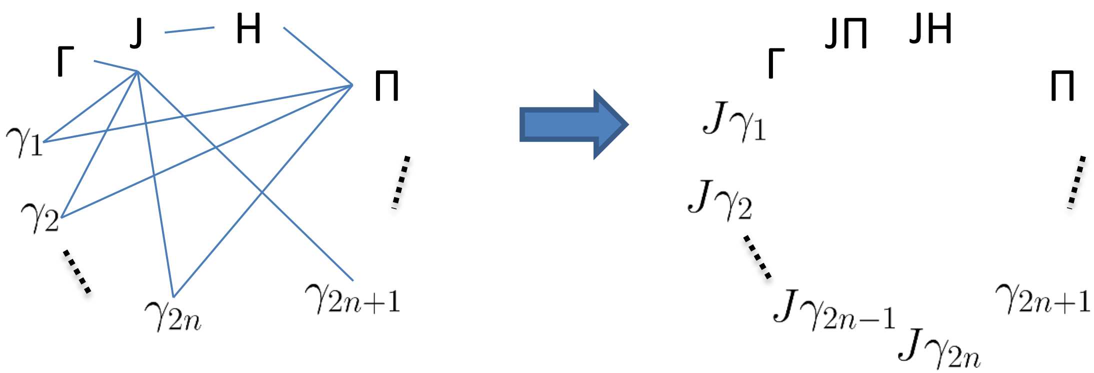


$$\Pi^2 = 1 \ : \ Cl_{2n-1,d-2n+3} \to Cl_{2n,d-2n+3} \qquad R_{4n-d-2}$$

$$\Pi^2 = -1 \ : \ Cl_{2n+1,d-2n+1} \to Cl_{2n+2,d-2n+1} \qquad R_{4n-d+2}$$

Complex, class AIII →半線形な対称性によりRealに

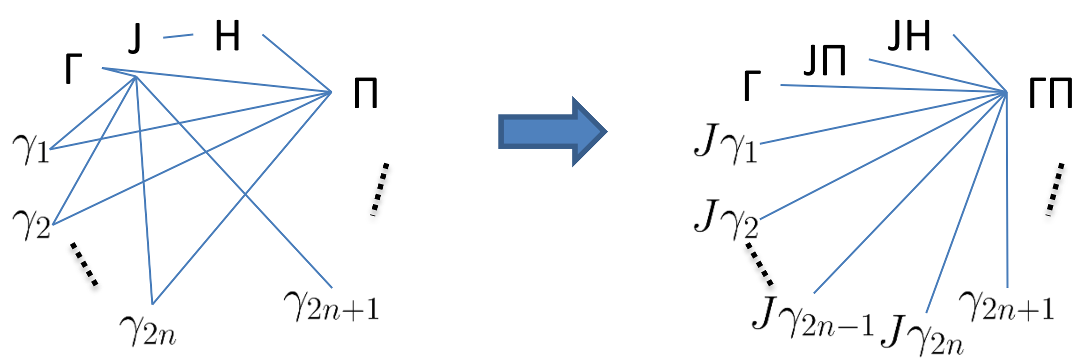
Π- : {Γ,Π}=0



$$\Pi^2 = 1 \ : \ Cl_{2n-1,d-2n+4} \rightarrow Cl_{2n,d-2n+4} \qquad R_{4n-d-3}$$

$$\Pi^2 = -1 \ : \ Cl_{2n+1,d-2n+2} \rightarrow Cl_{2n+2,d-2n+2} \qquad R_{4n-d+1}$$

Π+ : [Γ,Π]=0

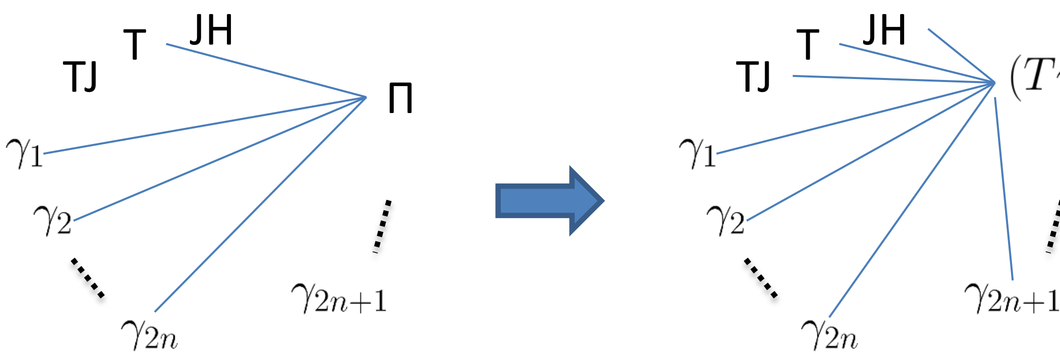


$$\Pi^2 = 1 \ : decouple, \ Cl_{2n-1,d-2n+3} \rightarrow Cl_{2n,d-2n+3} \qquad R_{4n-d-2}$$

$$\Pi^2 = -1 \ : complex, \ Cl_{d+2} \rightarrow Cl_{d+3} \qquad C_{d+2}$$

Real, class AI, AII

Π+ : [T,Π]=0

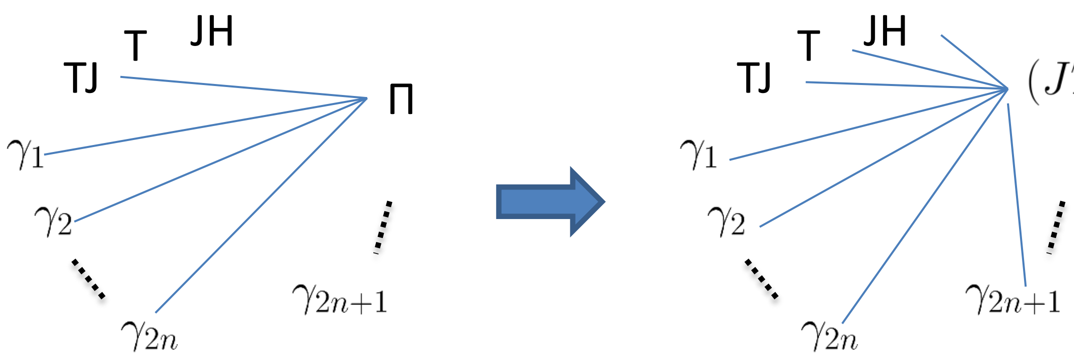


$$(T\gamma_1\cdots\gamma_{2n})\Pi = \tilde{\Pi}$$

$$\tilde{\Pi}^2 = \epsilon_T(-1)^n\Pi^2$$

$$\tilde{\Pi}^2 = 1 \;\rightarrow decouple$$
$$\tilde{\Pi}^2 = -1 \;\rightarrow complex$$

Π- : {T,Π}=0



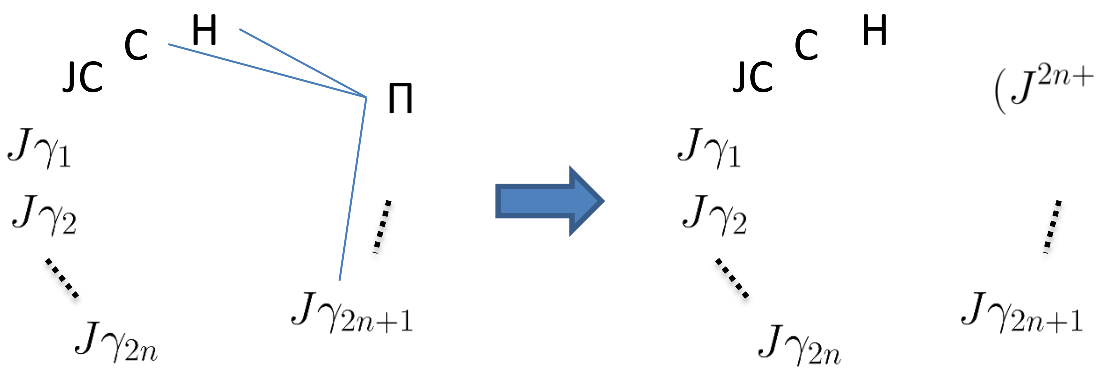
$$(JT\gamma_1\cdots\gamma_{2n})\Pi = \tilde{\Pi}$$

$$\tilde{\Pi}^2 = -\epsilon_T(-1)^n\Pi^2$$

$$\tilde{\Pi}^2 = 1 \;\rightarrow decouple$$
$$\tilde{\Pi}^2 = -1 \;\rightarrow complex$$

Real, class D, C

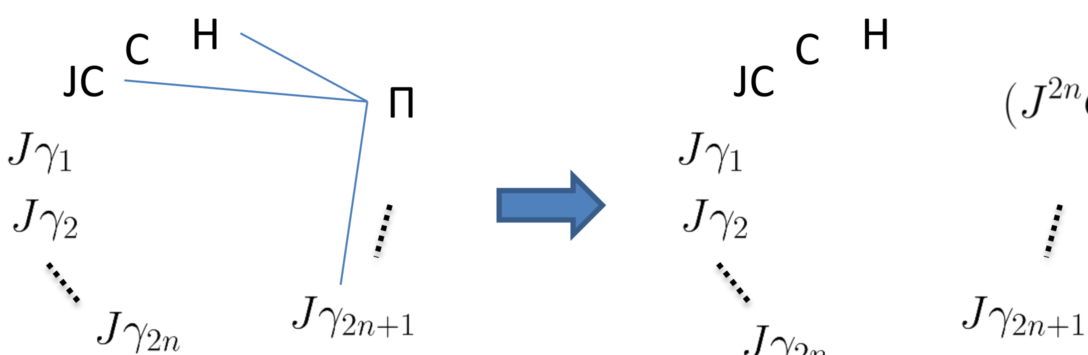
Π+ : [C,Π]=0



$$(J^{2n+1}C\gamma_1\cdots\gamma_{2n})\Pi = \tilde{\Pi}$$

$$\tilde{\Pi}^2 = -\epsilon_C(-1)^n\Pi^2$$

Π- : {C,Π}=0

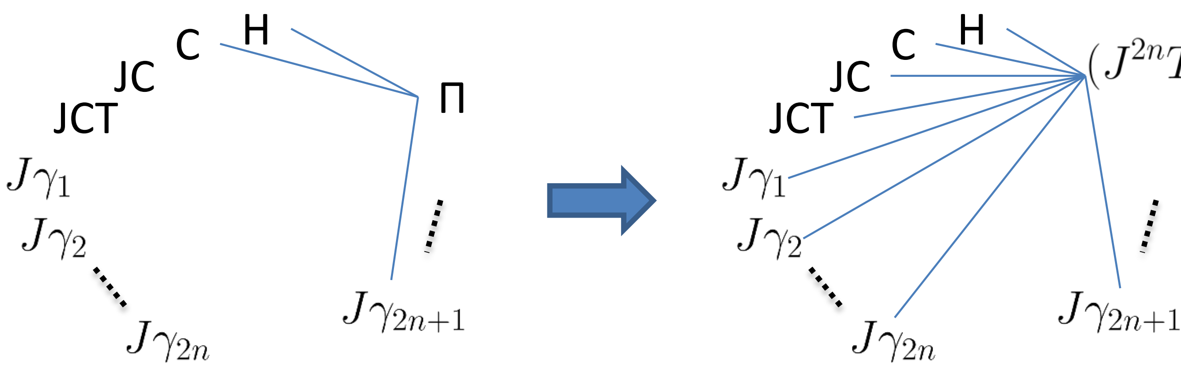


$$(J^{2n}C\gamma_1\cdots\gamma_{2n})\Pi = \tilde{\Pi}$$

$$\tilde{\Pi}^2 = -\epsilon_C(-1)^n\Pi^2$$

Real, class BDI, DIII, CII, CI

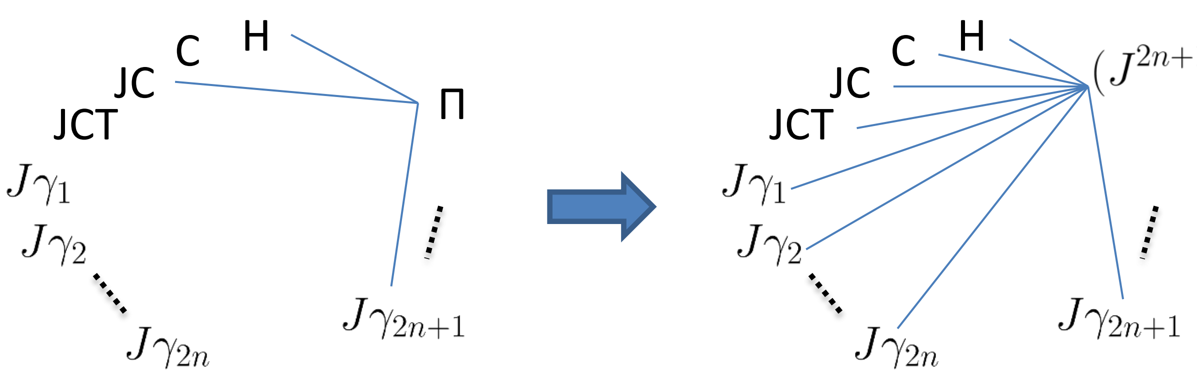
Π++ : [T,Π]=[C,Π]=0



$$\tilde{\Pi}^2 = \epsilon_T(-1)^n\Pi^2$$

$$\tilde{\Pi}^2 = 1 \ \rightarrow decouple$$
$$\tilde{\Pi}^2 = -1 \ \rightarrow complex$$

Π-- : {T,Π}={C,Π}=0

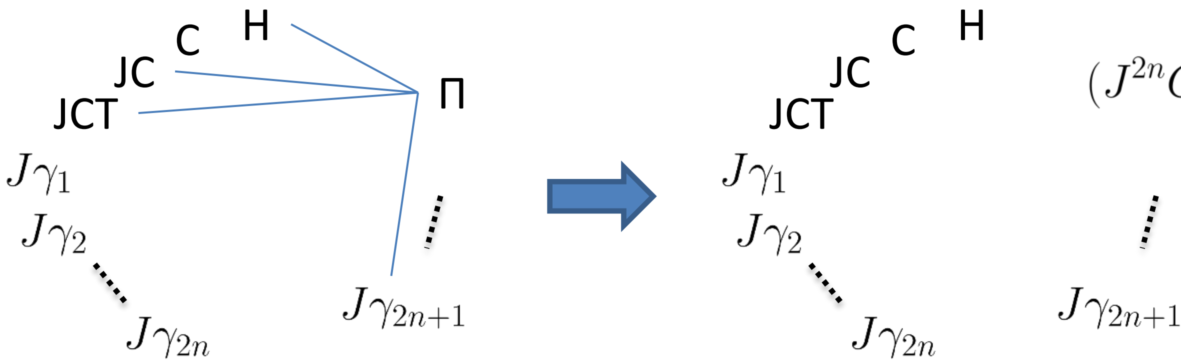


$$\tilde{\Pi}^2 = \epsilon_T(-1)^n\Pi^2$$

$$\tilde{\Pi}^2 = 1 \ \rightarrow decouple$$
$$\tilde{\Pi}^2 = -1 \ \rightarrow complex$$

Real, class BDI, DIII, CII, CI

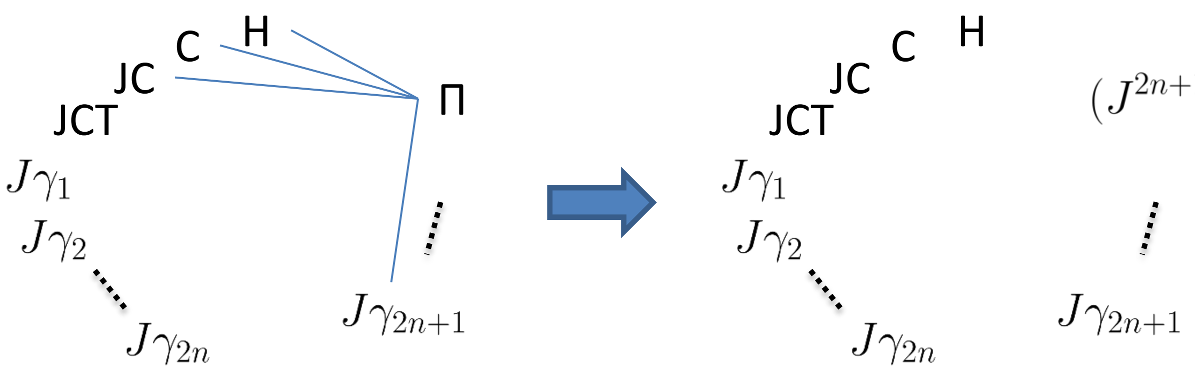
Π+- : [T,Π]={C,Π}=0



$$(J^{2n}C\gamma_1\cdots\gamma_{2n})\Pi=\tilde{\Pi}$$

$$\tilde{\Pi}^2=-\epsilon_C(-1)^n\Pi^2$$

Π-+ : {T,Π}=[C,Π]=0



$$(J^{2n+1}CT\gamma_1\cdots\gamma_{2n})\Pi=\tilde{\Pi}$$

$$\tilde{\Pi}^2=-(-1)^n\epsilon_T\epsilon_C\Pi^2$$

[aΠ1] Classification table for anti-linear Π $(-1)^n\Pi^2=1$ $\Pi_{\eta_\Gamma},\ \Pi_{\eta_T},\ \Pi_{\eta_C},\ \Pi_{\eta_T\eta_C}$
反転しない波数が 2n 個

class	Reflection		Cq or Rq	d=0	d=1	d=2	d=3	d=4	d=5	d=6	d=7
A	Π	real	R6			Z		Z2	Z2	Z	
AIII	Π+, Π^2=1	decouple, real	R6			Z		Z2	Z2	Z	
	Π+, Π^2=-1	complex	C0	Z		Z		Z		Z	
	Π-	real	R5		Z		Z2	Z2	Z		
AI	Π+	decouple	R0	Z				Z		Z2	Z2
	Π-	complex	C0	Z		Z		Z		Z	
BDI	Π++, Π--	decouple	R1	Z2	Z				Z		Z2
	Π+-, Π-+	-1	R0	Z				Z		Z2	Z2
D	Π	-1	R1	Z2	Z				Z		Z2
DIII	Π++, Π--	complex	C1		Z		Z		Z		Z
	Π+-	-1	R2	Z2	Z2	Z				Z	
	Π-+	+1	R4	Z		Z2	Z2	Z			
AII	Π+	complex	C0	Z		Z		Z		Z	
	Π-	decouple	R4	Z		Z2	Z2	Z			
CII	Π++, Π--	complex	C1		Z		Z		Z		Z
	Π+-	+1	R6			Z		Z2	Z2	Z	
	Π-+	-1	R4	Z		Z2	Z2	Z			
C	Π	+1	R7				Z		Z2	Z2	Z
CI	Π++, Π--	decouple	R7				Z		Z2	Z2	Z
	Π+-, Π-+	+1	R0	Z				Z		Z2	Z2

[aΠ2] Classification table for anti-linear Π $(-1)^n\Pi^2=-1$ $\Pi_{\eta_\Gamma},\ \Pi_{\eta_T},\ \Pi_{\eta_C},\ \Pi_{\eta_T\eta_C}$
反転しない波数が 2n 個

class	Reflection		Cq or Rq	d=0	d=1	d=2	d=3	d=4	d=5	d=6	d=7
A	Π	real	R6			Z		Z2	Z2	Z	
AIII	Π+, Π^2=1	decouple, real	R6			Z		Z2	Z2	Z	
	Π+, Π^2=-1	complex	C0	Z		Z		Z		Z	
	Π-	real	R5		Z		Z2	Z2	Z		
AI	Π+	complex	C0	Z		Z		Z		Z	
	Π-	decouple	R0	Z				Z		Z2	Z2
BDI	Π++, Π--	complex	C1		Z		Z		Z		Z
	Π+-, Π-+	+1	R2	Z2	Z2	Z				Z	
D	Π	+1	R3		Z2	Z2	Z				Z
DIII	Π++, Π--	decouple	R3		Z2	Z2	Z				Z
	Π+-	+1	R4	Z		Z2	Z2	Z			
	Π-+	-1	R2	Z2	Z2	Z				Z	
AII	Π+	decouple	R4	Z		Z2	Z2	Z			
	Π-	complex	C0	Z		Z		Z		Z	
CII	Π++, Π--	decouple	R5		Z		Z2	Z2	Z		
	Π+-	-1	R4	Z		Z2	Z2	Z			
	Π-+	+1	R6			Z		Z2	Z2	Z	
C	Π	-1	R5		Z		Z2	Z2	Z		
CI	Π++, Π--	complex	C1		Z		Z		Z		Z
	Π+-, Π-+	-1	R6			Z		Z2	Z2	Z	

具体例

・・・の前に補足

● 消滅演算子と生成演算子に対する対称性変換の位相部分は互いに複素共役の関係
→超流動状態ではトポロジカル相の分類において対称性変換の位相部分（というよりも global U(1)位相変換）が意味を持つ

$$\hat{g}\psi_i(\boldsymbol{x})\hat{g}^{-1} = R(g)_{ij}\psi_j(R_g(\boldsymbol{x}))$$

$$\hat{g}\psi_i^\dagger(\boldsymbol{x})\hat{g}^{-1} = R(g)_{ij}^*\psi_j^\dagger(R_g(\boldsymbol{x}))$$

Nambu spinor

⇒

$$\hat{g}\Psi(\boldsymbol{x})\hat{g}^{-1} = \begin{pmatrix} R(g) & \\ & R^*(g) \end{pmatrix} \Psi(R_g(\boldsymbol{x}))$$

$$\Psi(\boldsymbol{x}) = \begin{pmatrix} \psi_i(\boldsymbol{x}) \\ \psi_i^\dagger(\boldsymbol{x}) \end{pmatrix}$$

位相はオーダーパラメタがgのどの表現に属するかで決まる。

$$\hat{\Delta}_{MF} = \frac{1}{2}\sum_{\boldsymbol{k}} c_{\boldsymbol{k}}^\dagger \Delta(\boldsymbol{k}) c_{-\boldsymbol{k}}^\dagger + h.c.$$

$$\hat{g}\hat{\Delta}_{MF}\hat{g}^{-1} = \hat{\Delta}_{MF} \quad \Leftrightarrow \quad R^\dagger(g)\Delta(\boldsymbol{k})R^*(g) = \Delta(R_g(\boldsymbol{k}))$$

（空間群だと一般にはR(g)はk-依存）

- Topological mirror insulator   class AII + mirror

$$\mathcal{H}(k_x, k_y, k_z) = m s_3 + v(k_x \sigma_2 - k_y \sigma_1) s_1 + v_z k_z s_2$$

σ : spin
s : orbital

TR tr. $\hat{T}\Psi(\boldsymbol{k})\hat{T}^{-1} = i\sigma_2 \Psi(-\boldsymbol{k}),\ \hat{T} i \hat{T}^{-1} = -i\ \to T = i\sigma_2 \mathcal{K}$

yz面でのmirror (= inversion + x軸回りのπ回転)

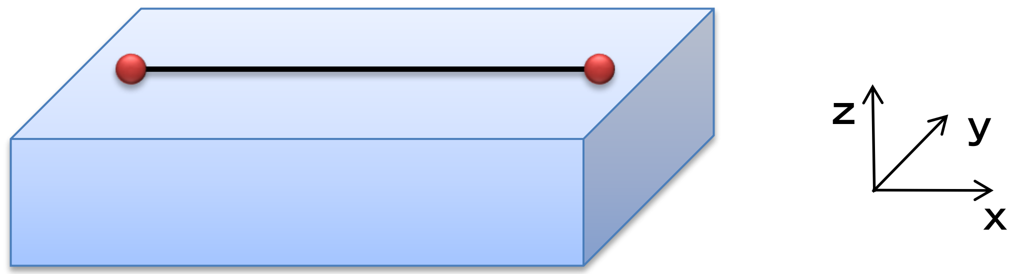
$$\hat{R}\psi(x,y,z)\hat{R}^{-1} = -i\sigma_1 \psi(-x,y,z)\ \to R = -i\sigma_1$$

$$\gamma_1 = \sigma_2 s_1,\ \gamma_2 = -\sigma_1 s_1,\ \gamma_3 = s_2,\ J = i,\ T = i\sigma_2 \mathcal{K},\ R = -i\sigma_1$$

class AII + reflection symmetry R : (x,y,z) → (-x,y,z)

$R^2 = -1,\ \ [T, R] = 0 \to R_+$ ⇒ Z (table R1)

- 1D nanowire + s-wave SC + Zeeman class D × (zx面mirror + TR)



$$H_{MF} = \frac{1}{2}\sum_{\boldsymbol{k}} \Psi^\dagger_{\boldsymbol{k}} \mathcal{H}(\boldsymbol{k}) \Psi_{\boldsymbol{k}} \qquad \Psi = (\psi_\uparrow, \psi_\downarrow, \psi^\dagger_\uparrow, \psi^\dagger_\downarrow)^T$$

$$\mathcal{H}(k_x) = \begin{pmatrix} \frac{k_x^2}{2m} - \mu + k_x\sigma_y + \boldsymbol{h}\cdot\boldsymbol{\sigma} & \Delta i\sigma_2 \\ -i\sigma_2\Delta & -\frac{k_x^2}{2m} + \mu - k_x\sigma_y - \boldsymbol{h}\cdot\boldsymbol{\sigma} \end{pmatrix}$$

$$= (\frac{k_x^2}{2m} - \mu)\tau_z + k_x\sigma_y\tau_z - \Delta\sigma_y\tau_y + \boldsymbol{h}\cdot\boldsymbol{\sigma}\tau_z$$

PH tr. $\hat{C}\Psi(\boldsymbol{k})\hat{C}^{-1} = \tau_1\Psi(-\boldsymbol{k}),\ \hat{C}i\hat{C}^{-1} = -i \ \rightarrow C = \tau_1\mathcal{K}$

TR tr. $\hat{T}\Psi(\boldsymbol{k})\hat{T}^{-1} = i\sigma_2\Psi(-\boldsymbol{k}),\ \hat{T}i\hat{T}^{-1} = -i \ \rightarrow T = i\sigma_2\mathcal{K}$

zx面でのmirror (= inversion + y軸回りのπ回転) $\hat{M}_{zx}\Psi(k_x,k_y,k_z)\hat{M}_{zx}^{-1} = -i\sigma_y\Psi(k_x,-k_y,k_z)$

$$\hat{S} := \hat{T}\hat{M}_{zx} \ : \ \hat{S}\Psi(k_x,k_y,k_z)\hat{S}^{-1} = \Psi(-k_x,k_y,-k_z),\ \hat{S}i\hat{S}^{-1} = -i \rightarrow S = \mathcal{K}$$

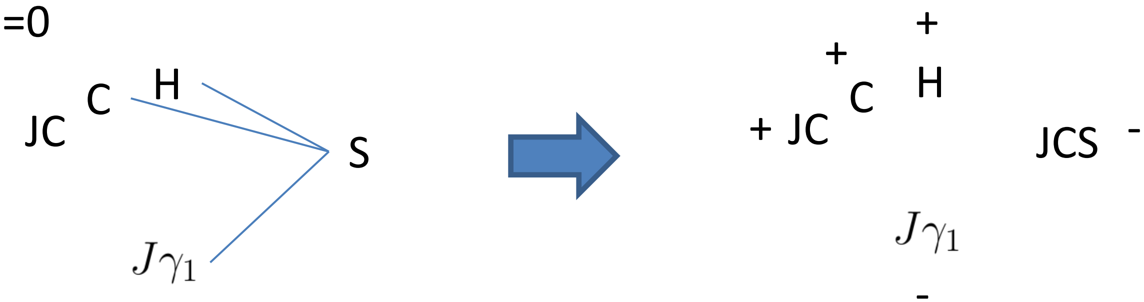
$$\mathcal{H}(k_x) = \begin{pmatrix} \frac{k_x^2}{2m} - \mu + k_x\sigma_y + \boldsymbol{h}\cdot\boldsymbol{\sigma} & \Delta i\sigma_2 \\ -i\sigma_2\Delta & -\frac{k_x^2}{2m} + \mu - k_x\sigma_y - \boldsymbol{h}\cdot\boldsymbol{\sigma} \end{pmatrix}$$
$$= (\frac{k_x^2}{2m} - \mu)\tau_z + k_x\sigma_y\tau_z - \Delta\sigma_y\tau_y + \boldsymbol{h}\cdot\boldsymbol{\sigma}\tau_z$$

PH tr. $\hat{C}\Psi(\boldsymbol{k})\hat{C}^{-1} = \tau_1\Psi(-\boldsymbol{k}),\ \hat{C}i\hat{C}^{-1} = -i \ \rightarrow C = \tau_1\mathcal{K}$

$$\hat{S} := \hat{T}\hat{M}_{zx} \ : \ \hat{S}\Psi(k_x,k_y,k_z)\hat{S}^{-1} = \Psi(-k_x,k_y,-k_z),\ \hat{S}i\hat{S}^{-1} = -i \rightarrow S = \mathcal{K}$$

$$C = \tau_x\mathcal{K}, J = i, \gamma_1 = \sigma_y\tau_z, S = \mathcal{K}$$

[C,S]=0



$$Cl_{2,2} \rightarrow Cl_{2,3} \ : \ \pi_0(R_0) = \mathbb{Z}$$

(odd parity)でも同じ結果

$$\mathcal{H}(k_x) = \begin{pmatrix} \frac{k_x^2}{2m} - \mu + \boldsymbol{h}\cdot\boldsymbol{\sigma} & \frac{\Delta_p}{k_F} k_x \sigma_x (i\sigma_y) \\ -i\sigma_y \frac{\Delta}{k_F} k_x \sigma_x & -\frac{k_x^2}{2m} + \mu - \boldsymbol{h}\cdot\boldsymbol{\sigma} \end{pmatrix}$$

$$= (\frac{k_x^2}{2m} - \mu)\tau_z + \frac{\Delta}{k_F} k_x \sigma_z \tau_x + \boldsymbol{h}\cdot\boldsymbol{\sigma}\tau_z$$

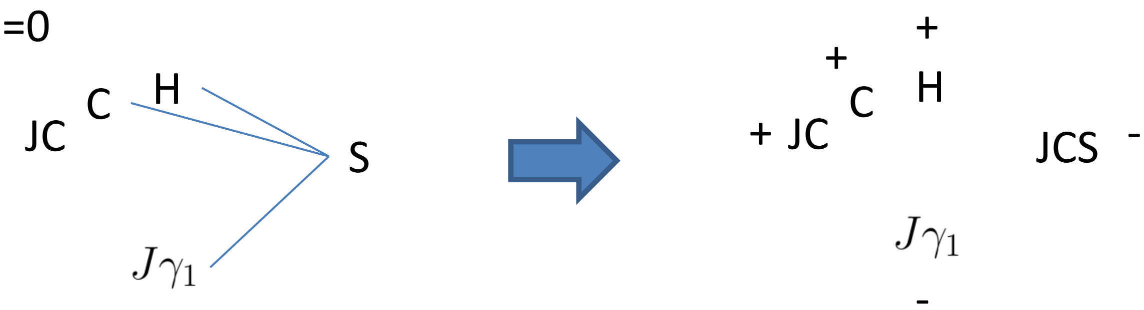
←トポロジカルなモデルではないかも

PH tr. $\hat{C}\Psi(\boldsymbol{k})\hat{C}^{-1} = \tau_1 \Psi(-\boldsymbol{k}), \ \hat{C} i \hat{C}^{-1} = -i \ \ \rightarrow C = \tau_1 \mathcal{K}$

$$\hat{S} := U(\pi/2)\hat{T}\hat{M}_{zx} \ : \ \hat{S}\Psi(k_x, k_y, k_z)\hat{S}^{-1} = i\tau_z \Psi(-k_x, k_y, -k_z), \ \hat{S} i \hat{S}^{-1} = -i \rightarrow S = i\tau_z \mathcal{K}$$

$$C = \tau_x \mathcal{K}, J = i, \gamma_1 = \sigma_z \tau_x, S = i\tau_z \mathcal{K}$$

[C,S]=0



$$Cl_{2,2} \rightarrow Cl_{2,3} \ : \ \pi_0(R_0) = \mathbb{Z}$$

- Topological mirror superconductivity　　　class D + mirror

$$H_{MF} = \frac{1}{2}\sum \Psi^\dagger_{\boldsymbol{k}} \mathcal{H}(\boldsymbol{k}) \Psi_{\boldsymbol{k}} \qquad \Psi = (\psi_\uparrow, \psi_\downarrow, \psi^\dagger_\uparrow, \psi^\dagger_\downarrow)^T$$

$$\mathcal{H}(k_x,k_y,k_z) = \begin{pmatrix} \varepsilon(\boldsymbol{k}) - \mu + h_z\sigma_3 & \frac{\Delta_p}{k_F}\boldsymbol{k}\cdot\boldsymbol{\sigma}(i\sigma_2) \\ (-i\sigma_2)\frac{\Delta_p}{k_F}\boldsymbol{k}\cdot\boldsymbol{\sigma} & -\varepsilon^T(-\boldsymbol{k}) + \mu - h_z\sigma_3 \end{pmatrix}$$

$$= \tau_3(\varepsilon(\boldsymbol{k}) - \mu) + \frac{\Delta}{k_F}(-k_x\tau_1\sigma_3 - k_y\tau_2 + k_z\tau_1\sigma_1) + h_z\tau_3\sigma_3$$

PH tr. $\hat{C}\Psi(\boldsymbol{k})\hat{C}^{-1} = \tau_1\Psi(-\boldsymbol{k}),\ \hat{C}i\hat{C}^{-1} = -i \ \rightarrow C = \tau_1\mathcal{K}$

xy面でのmirror (= inversion + z軸回りのπ回転)

$$\hat{M}_{xy}\psi(x,y,z)\hat{M}_{xy}^{-1} = -i\sigma_3\psi(x,y,-z)$$
$$\hat{M}_{xy}\psi^\dagger(x,y,z)\hat{M}_{xy}^{-1} = i\sigma_3\psi^\dagger(x,y,-z)$$

global U(1) phase transformation

$$\hat{U}(\theta)\psi(x,y,z)\hat{U}(\theta)^{-1} = e^{i\theta}\psi(x,y,z)$$
$$\hat{U}(\theta)\psi^\dagger(x,y,z)\hat{U}(\theta)^{-1} = e^{-i\theta}\psi^\dagger(x,y,z)$$

$\hat{R} := \hat{U}(\pi/2)\hat{M}_{xy}$

この位相の選び方は超伝導
オーダーで決まる

$$\hat{R}\psi(x,y,z)\hat{R}^{-1} = \sigma_3\psi(x,y,-z)$$
$$\hat{R}\psi^\dagger(x,y,z)\hat{R}^{-1} = \sigma_3\psi^\dagger(x,y,-z)$$
$$\hat{R}\Psi(k_x,k_y,k_z)\hat{R}^{-1} = \sigma_3\Psi(k_x,k_y,-k_z) \ \rightarrow R = \sigma_3$$

$$\gamma_1 = -\tau_1\sigma_3,\ \gamma_2 = -\tau_2,\ \gamma_3 = \tau_1\sigma_1,\ J = i,\ C = \tau_1\mathcal{K},\ R = \sigma_3$$

class D + reflection symmetry R : (x,y,z) → (x,y,-z)

$R^2 = 1,\ \ [C,R] = 0 \rightarrow R_+$ ⇒ Z (table R2)

● He-B + 面内の磁場　　class D + (π-rotation　×　Time-reversal)

Mizushima-Sato-Machida, PRL (2012).

$$H_{MF}=\frac{1}{2}\sum_{\boldsymbol{k}}\Psi_{\boldsymbol{k}}^{\dagger}\mathcal{H}(\boldsymbol{k})\Psi_{\boldsymbol{k}} \qquad \Psi=(\psi_\uparrow,\psi_\downarrow,\psi_\uparrow^\dagger,\psi_\downarrow^\dagger)^T$$

$$\mathcal{H}(k_x,k_y,k_z)=\begin{pmatrix}\varepsilon(\boldsymbol{k})-\mu+h_x\sigma_1+h_y\sigma_2 & \frac{\Delta_p}{k_F}\boldsymbol{k}\cdot\boldsymbol{\sigma}(i\sigma_2)\\ (-i\sigma_2)\frac{\Delta_p}{k_F}\boldsymbol{k}\cdot\boldsymbol{\sigma} & -\varepsilon^T(-\boldsymbol{k})+\mu-h_x\sigma_1-h_y\sigma_2\end{pmatrix}$$

$$=\tau_3(\varepsilon(\boldsymbol{k})-\mu)+\frac{\Delta}{k_F}(-k_x\tau_1\sigma_3-k_y\tau_2+k_z\tau_1\sigma_1)+h_x\tau_3\sigma_1+h_y\tau_3\sigma_1$$

PH tr. $\hat{C}\Psi(\boldsymbol{k})\hat{C}^{-1}=\tau_1\Psi(-\boldsymbol{k}),\ \hat{C}i\hat{C}^{-1}=-i\ \rightarrow C=\tau_1\mathcal{K}$

TR tr. $\hat{T}\Psi(\boldsymbol{k})\hat{T}^{-1}=i\sigma_2\Psi(-\boldsymbol{k}),\ \hat{T}i\hat{T}^{-1}=-i\ \rightarrow T=i\sigma_2\mathcal{K}$

xy面内SO(2)L+S のπ-rotation $\hat{U}(\pi)\Psi(x,y,z)\hat{U}(\pi)^{-1}=-i\sigma_3\tau_3\Psi(-x,-y,z)$

$$\hat{R}:=\hat{T}\hat{U}(\pi) \quad \hat{R}\Psi(x,y,z)\hat{R}^{-1}=i\sigma_1\tau_3\Psi(-x,-y,z),\ \hat{R}i\hat{R}^{-1}=-i\ \rightarrow R=i\sigma_1\tau_3\mathcal{K}$$

$$\gamma_1=-\tau_1\sigma_3,\ \gamma_2=-\tau_2,\ \gamma_3=\tau_1\sigma_1,\ J=i,\ C=\tau_1\mathcal{K},\ R=i\sigma_1\tau_3\mathcal{K}$$

class D + anti-unitary symmetry (x,y,z) → (-x,-y,z)

$R^2=1,\ \ [C,R]=0\rightarrow R_+$ ⇒ Z (table Π2)

- Topological inversion superconductivity   class D + inversion

$$H_{MF} = \frac{1}{2}\sum_{\boldsymbol{k}} \Psi_{\boldsymbol{k}}^\dagger \mathcal{H}(\boldsymbol{k})\Psi_{\boldsymbol{k}} \qquad \Psi = (\psi_\uparrow, \psi_\downarrow, \psi_\uparrow^\dagger, \psi_\downarrow^\dagger)^T$$

$$\mathcal{H}(k_x, k_y, k_z) = \begin{pmatrix} \varepsilon(\boldsymbol{k}) - \mu + h_x\sigma_1 + h_y\sigma_2 + h_z\sigma_3 & \frac{\Delta_p}{k_F}\boldsymbol{k}\cdot\boldsymbol{\sigma}(i\sigma_2) \\ (-i\sigma_2)\frac{\Delta_p}{k_F}\boldsymbol{k}\cdot\boldsymbol{\sigma} & -\varepsilon^T(-\boldsymbol{k}) + \mu - h_x\sigma_1 - h_y\sigma_2 - h_z\sigma_3 \end{pmatrix}$$

$$= \tau_3(\varepsilon(\boldsymbol{k}) - \mu) + \frac{\Delta}{k_F}(-k_x\tau_1\sigma_3 - k_y\tau_2 + k_z\tau_1\sigma_1) + h_x\tau_3\sigma_1 + h_y\tau_3\sigma_2 + h_z\tau_3\sigma_3$$

PH tr. $\hat{C}\Psi(\boldsymbol{k})\hat{C}^{-1} = \tau_1\Psi(-\boldsymbol{k}),\ \hat{C}i\hat{C}^{-1} = -i \ \rightarrow C = \tau_1\mathcal{K}$

Inversion + global π/2 phase rotation (odd parity SC)

$$\hat{R}\Psi(x,y,z)\hat{R}^{-1} = i\tau_3\Psi(-x,-y,-z) \ \rightarrow R = i\tau_3$$

$$\gamma_1 = -\tau_1\sigma_3,\ \gamma_2 = -\tau_2,\ \gamma_3 = \tau_1\sigma_1,\ J = i,\ C = \tau_1\mathcal{K},\ R = i\tau_3$$

class D + unitary symmetry R : (x,y,z) → (-x,-y,-z)

$R^2 = -1,\quad [C, R] = 0 \rightarrow R_+$ ⇒ Z (table I2)

Odd parity superconductors are always topological ?

Additional symmetry + defect zero mode

Defectを囲むパラメタに対するreflection, π-rotationの作用に仕方を確認しよう

とりあえず２次元、３次元で

Defectを囲むパラメタの共通する交換関係（再掲）

D : co-dimension of defect

$$H_{d,D} = \sum_{i=1}^{d} \gamma_i k_i + \sum_{a=1}^{D} \tilde{\gamma}_a s_a + H$$

$$\gamma_i^2 = \tilde{\gamma}_a^2 = 1$$

$$\{H, \Gamma\} = [H, J] = [H, T] = \{H, C\} = \{J, T\} = \{J, C\} = [T, C] = 0$$

$$\{\gamma_i, H\} = [\gamma_i, J] = \{\gamma_i, T\} = [\gamma_i, C] = \{\gamma_i, \Gamma\} = 0$$

$$\{\tilde{\gamma}_a, H\} = [\tilde{\gamma}_a, J] = [\tilde{\gamma}_a, T] = \{\tilde{\gamma}_a, C\} = \{\tilde{\gamma}_a, \Gamma\} = 0$$

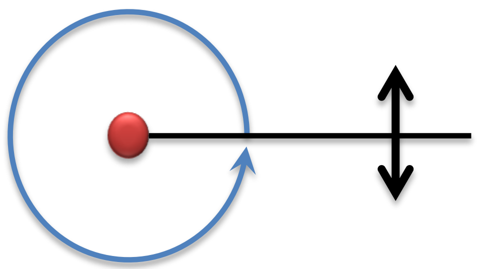
or

$$\{J\tilde{\gamma}_a, H\} = [J\tilde{\gamma}_a, J] = \{J\tilde{\gamma}_a, T\} = [J\tilde{\gamma}_a, C] = \{J\tilde{\gamma}_a, \Gamma\} = 0$$

$$(J\tilde{\gamma}_a)^2 = -1$$

2-dim. Point defect

- reflection



$$\phi \to -\phi$$

$$\{R, \tilde{\gamma}_\phi\} = 0$$

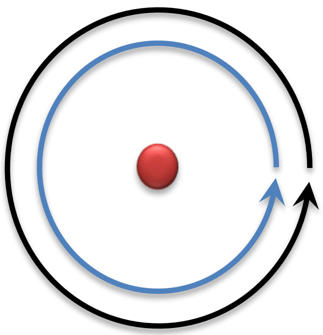
R : unitary $\to \{R, J\gamma_\phi\} = 0$

→反転する波数がひとつ増える, (J γ)^2= -1

R : antiunitary $\to [R, J\gamma_\phi] = 0$

→反転しない波数がひとつ増える, (J γ)^2= -1

- π-rotation



$$\phi \to \phi + \pi \sim \phi$$

$$[\Pi, \tilde{\gamma}_\phi] = 0$$

Π : unitary $\to [\Pi, J\gamma_\phi] = 0$

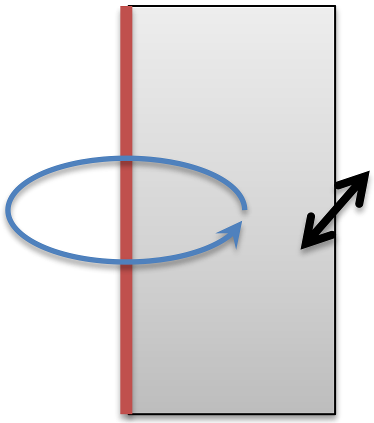
→反転しない波数がひとつ増える, (J γ)^2= -1

Π : antiunitary $\to \{\Pi, J\gamma_\phi\} = 0$

→反転する波数がひとつ増える, (J γ)^2= -1

3-dim. line defect

- Reflection (mirror plane // line defect)



$$\phi \to -\phi$$

$$\{R, \tilde{\gamma}_\phi\} = 0$$

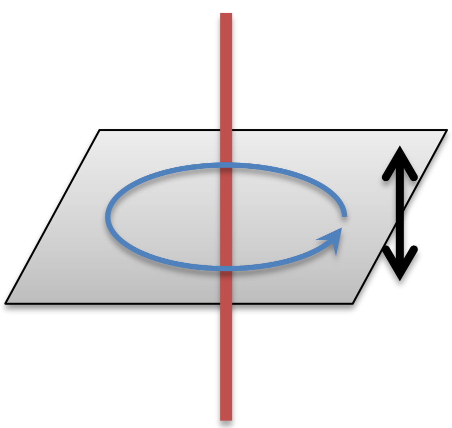
$$R \;:\; \text{unitary} \;\to \{R, J\gamma_\phi\} = 0$$

→反転する波数がひとつ増える, (J γ)^2= -1

$$R \;:\; \text{antiunitary} \;\to [R, J\gamma_\phi] = 0$$

→反転しない波数がひとつ増える, (J γ)^2= -1

- Reflection (mirror plane ⊥ line defect)



$$\phi \to \phi$$

$$[R, \tilde{\gamma}_\phi] = 0$$

$$R \;:\; \text{unitary} \;\to [R, J\gamma_\phi] = 0$$

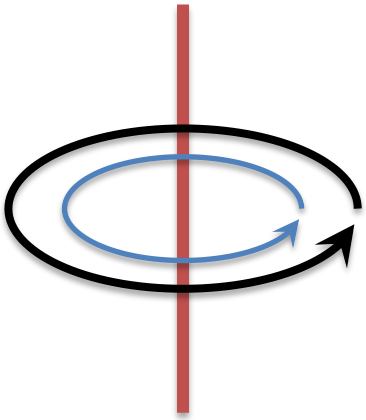
→反転しない波数がひとつ増える, (J γ)^2= -1

$$R \;:\; \text{antiunitary} \;\to \{R, J\gamma_\phi\} = 0$$

→反転する波数がひとつ増える, (J γ)^2= -1

3-dim. line defect

- π-rotation (rotation axis // line defect)



$$\phi \to \phi + \pi \sim \phi$$

$$[\Pi, \tilde{\gamma}_\phi] = 0$$

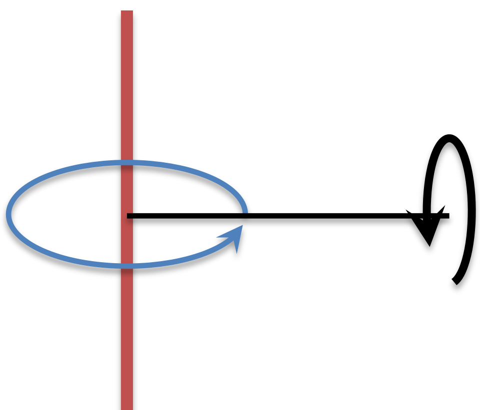
$$\Pi \ : \ \text{unitary} \ \to [\Pi, J\gamma_\phi] = 0$$

→反転しない波数がひとつ増える, (J γ)^2= -1

$$\Pi \ : \ \text{antiunitary} \ \to \{\Pi, J\gamma_\phi\} = 0$$

→反転する波数がひとつ増える, (J γ)^2= -1

- π-rotation (rotation axis ⊥ line defect)



$$\phi \to -\phi$$

$$\{\Pi, \tilde{\gamma}_\phi\} = 0$$

$$\Pi \ : \ \text{unitary} \ \to \{\Pi, J\gamma_\phi\} = 0$$

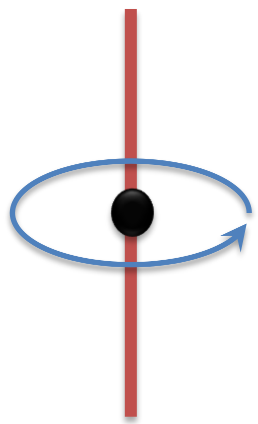
→反転する波数がひとつ増える, (J γ)^2= -1

$$\Pi \ : \ \text{antiunitary} \ \to [\Pi, J\gamma_\phi] = 0$$

→反転しない波数がひとつ増える, (J γ)^2= -1

3-dim. line defect

- inversion



$$\phi \to \phi + \pi \sim \phi$$

$$[R, \tilde{\gamma}_\phi] = 0$$

R : unitary $\to [R, J\gamma_\phi] = 0$

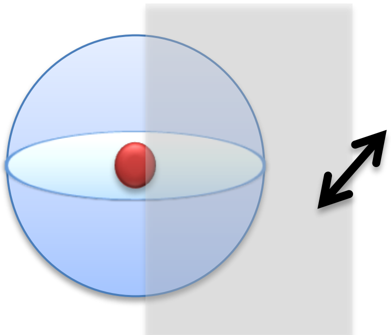
→反転しない波数がひとつ増える, (J γ)^2= -1

R : antiunitary $\to \{R, J\gamma_\phi\} = 0$

→反転する波数がひとつ増える, (J γ)^2= -1

3-dim. Point defect

- reflection



$$\theta \to \theta \qquad [R, \tilde{\gamma}_\theta] = 0$$

$$\phi \to -\phi \qquad \{R, \tilde{\gamma}_\phi\} = 0$$

$$R \ : \ \text{unitary} \ \to [R, J\gamma_\theta] = 0, \ \{R, J\gamma_\phi\} = 0$$

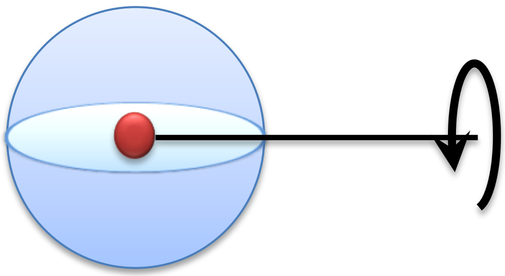
→反転する波数と反転しない波数がそれぞれひとつ増える, (J γ)^2= -1

$$R \ : \ \text{antiunitary} \ \to \{R, J\gamma_\theta\} = 0, \ [R, J\gamma_\phi] = 0$$

→反転する波数と反転しない波数がそれぞれひとつ増える, (J γ)^2= -1

3-dim. Point defect

- π-rotation



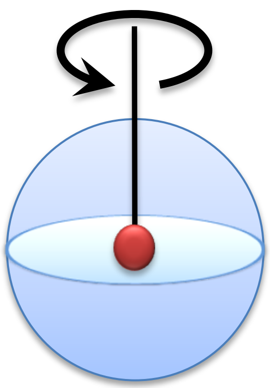
$$\theta \to \pi - \theta \sim -\theta \qquad \{\Pi, \tilde{\gamma}_\theta\} = 0$$
$$\phi \to -\phi \qquad \{\Pi, \tilde{\gamma}_\phi\} = 0$$
$$\Pi \;:\; \text{unitary} \;\to \{\Pi, J\gamma_\theta\} = 0,\; \{\Pi, J\gamma_\phi\} = 0$$

→反転する波数が2つ増える, (J γ)^2= -1

$$\Pi \;:\; \text{antiunitary} \;\to [\Pi, J\gamma_\theta] = 0,\; [\Pi, J\gamma_\phi] = 0$$

→反転しない波数が2つ増える, (J γ)^2= -1

⇕ 不定性あり



$$\theta \to \theta \qquad [\Pi, \tilde{\gamma}_\theta] = 0$$
$$\phi \to \phi + \pi \sim \phi \qquad [\Pi, \tilde{\gamma}_\phi] = 0$$
$$\Pi \;:\; \text{unitary} \;\to [\Pi, J\gamma_\theta] = 0,\; [\Pi, J\gamma_\phi] = 0$$

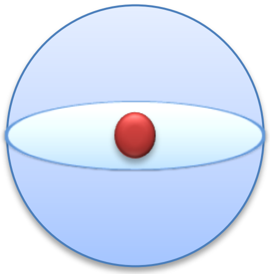
→反転しない波数が2つ増える, (J γ)^2= -1

$$\Pi \;:\; \text{antiunitary} \;\to \{\Pi, J\gamma_\theta\} = 0,\; \{\Pi, J\gamma_\phi\} = 0$$

→反転する波数が2つ増える, (J γ)^2= -1

3-dim. Point defect

- inversion



$$\theta \to \pi - \theta \sim -\theta \qquad \{R, \tilde{\gamma}_\theta\} = 0$$

$$\phi \to \phi + \pi \sim \phi \qquad [R, \tilde{\gamma}_\phi] = 0$$

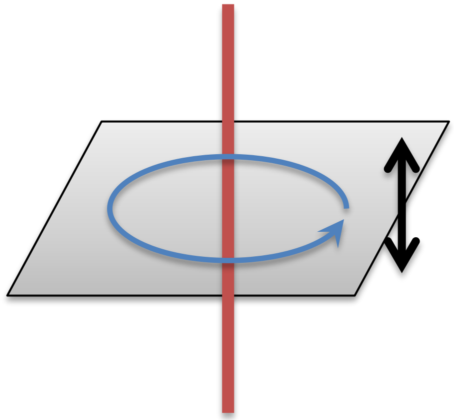
$$R \ : \ \text{unitary} \ \to \{R, J\gamma_\theta\} = 0, \ [R, J\gamma_\phi] = 0$$

→反転する波数と反転しない波数がそれぞれひとつ増える, (J γ)^2= -1

$$R \ : \ \text{antiunitary} \ \to [R, J\gamma_\theta] = 0, \ \{R, J\gamma_\phi\} = 0$$

→反転する波数と反転しない波数がそれぞれひとつ増える, (J γ)^2= -1

(e.g.) vortex line in class D + unitary reflection



$$[R, \tilde{\gamma}_\phi] = 0$$

$$\{R, \gamma_3\} = 0$$

R+ : [C,R]=0

H

C

$J\gamma_3 R$

JC

$\tilde{\gamma}_1$

$J\gamma_1$

$J\gamma_3$

$J\gamma_2$

$$\text{class D},\ R^2 = 1\ :\ Cl_{3,4} \to Cl_{3,5}\ \ : R_1,\ \pi_0(R_1) = \mathbb{Z}_2$$

$$\text{class D},\ R^2 = -1\ :\ Cl_{4,3} \to Cl_{4,4}\ \ : R_7,\ \pi_0(R_7) = 0$$

R- : {C,R}=0

H

C

$\gamma_3 R$

JC

$\tilde{\gamma}_1$

$J\gamma_1$

$J\gamma_3$

$J\gamma_2$

$$\text{class D},\ R^2 = 1\ :\ Cl_{4,3} \to Cl_{4,4}\ \ : R_7,\ \pi_0(R_7) = 0$$

$$\text{class D},\ R^2 = -1\ :\ Cl_{3,4} \to Cl_{3,5}\ \ : R_1,\ \pi_0(R_1) = \mathbb{Z}_2$$

Classification of defect zero mode with a unitary/anti-unitary reflection symmetry

$$R\ :\ (x_1, x_2, x_3, \dots) \to (-x_1, x_2, x_3, \dots)$$

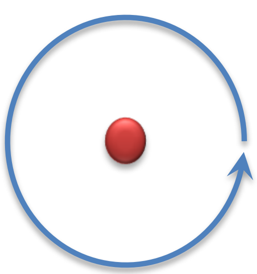
座標をひとつ反転

任意次元のdefectを考える

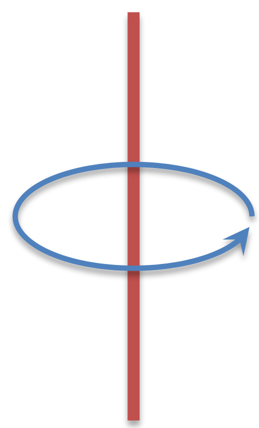
- point defect (δ-1 = 0)
- line defect (δ-1 = 1)
- membrane defect (δ-1 = 2)
- ・・・

d : dimension
δ-1 : defect dimension
D : co-dimension of defect
δ = d - D

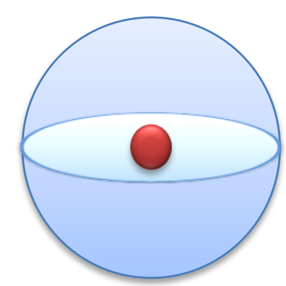
$$(s_1, s_2, \ldots, s_D)$$: parameters surrounding a defect



$d = 2, \delta - 1 = 0, (s_1)$



$d = 3, \delta - 1 = 1, (s_1)$



$d = 3, \delta - 1 = 0, (s_1, s_2)$

D : co-dimension of defect

$$H_{d,D} = \sum_{i=1}^{d} \gamma_i k_i + \sum_{a=1}^{D} \tilde{\gamma}_a s_a + H$$

$$\gamma_i^2 = \tilde{\gamma}_a^2 = 1$$

$$\{H, \Gamma\} = [H, J] = [H, T] = \{H, C\} = \{J, T\} = \{J, C\} = [T, C] = 0$$

$$\{\gamma_i, H\} = [\gamma_i, J] = \{\gamma_i, T\} = [\gamma_i, C] = \{\gamma_i, \Gamma\} = 0$$

$$\{\tilde{\gamma}_a, H\} = [\tilde{\gamma}_a, J] = [\tilde{\gamma}_a, T] = \{\tilde{\gamma}_a, C\} = \{\tilde{\gamma}_a, \Gamma\} = 0$$

D次元球面のパラメタ付け

$$\hat{\boldsymbol{n}} = (n_1, n_2, \ldots, n_{D+1})$$

$$n_1 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4 \cdots \sin\theta_D$$

$$n_2 = \cos\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4 \cdots \sin\theta_D$$

$$n_3 = \cos\theta_2 \sin\theta_3 \sin\theta_4 \cdots \sin\theta_D$$

$$n_4 = \cos\theta_3 \sin\theta_4 \cdots \sin\theta_D$$

$$n_5 = \cos\theta_4 \cdots \sin\theta_D$$

$$\cdots$$

$$n_D = \cos\theta_{D-1} \sin\theta_D$$

$$n_{D+1} = \cos\theta_D$$

$$(n_1, n_2, n_3, \ldots) \to (-n_1, n_2, n_3, \ldots) \;:\; \theta_1 \to -\theta_1$$

$$(n_1, n_2, n_3, \ldots) \to (n_1, -n_2, n_3, \ldots) \;:\; \theta_1 \to \pi - \theta_1$$

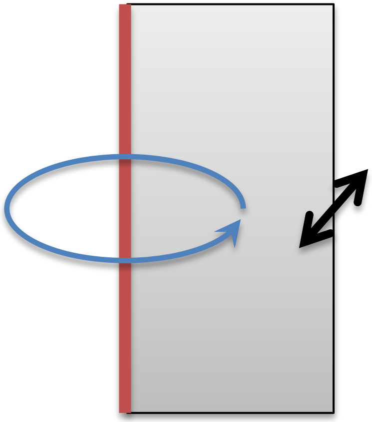
$$(n_1, n_2, n_3, \ldots) \to (n_1, n_2, -n_3, \ldots) \;:\; \theta_1 \to \pi - \theta_2$$

- defectを囲むパラメタへのreflectionの作用に仕方は2通りある

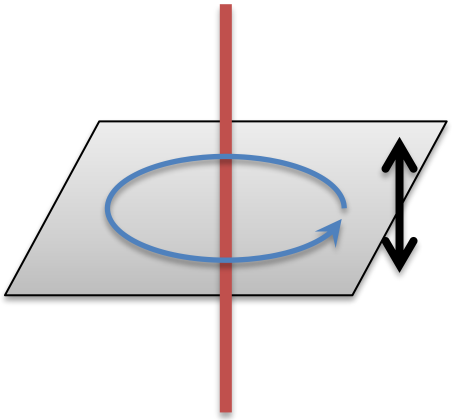
(I) reflectionを受ける座標が$(n_1, n_2, \ldots, n_{D+1})$に入っている
→ひとつだけ反転：$(s_1,s_2,\ldots,s_D) \to (-s_1,s_2,\ldots,s_D)$

(II) reflectionを受ける座標が$(n_1, n_2, \ldots, n_{D+1})$に入っていない
→反転しない：$(s_1,s_2,\ldots,s_D) \to (s_1,s_2,\ldots,s_D)$

- point defectはタイプ(I)のみ
- タイプ(ii)の特別な場合としてboundaryの分類(D=0)を含むことに注意



type I



type II

Unitary symmetry の場合は純虚数 J を用いていつでも $R^2 = 1$ にとれる
が、Jを付け加えることは生の対称性変換ではない場合があるので不自然のような・・・

Anti-unitary symmetry の場合は $R^2 = 1$, $R^2 = -1$ の両方の場合を考える必要がある？

Unitary reflection, type (I)

D : co-dimension of defect

$$H_{d,D} = \gamma_1 k_1 + \sum_{i=2}^{d} \gamma_i k_i + \tilde{\gamma}_1 s_1 + \sum_{a=2}^{D} \tilde{\gamma}_a s_a + H$$

$$\gamma_i^2 = \tilde{\gamma}_a^2 = 1$$

$$\{H, \Gamma\} = [H, J] = [H, T] = \{H, C\} = \{J, T\} = \{J, C\} = [T, C] = 0$$

$$\{\gamma_i, H\} = [\gamma_i, J] = \{\gamma_i, T\} = [\gamma_i, C] = \{\gamma_i, \Gamma\} = 0$$

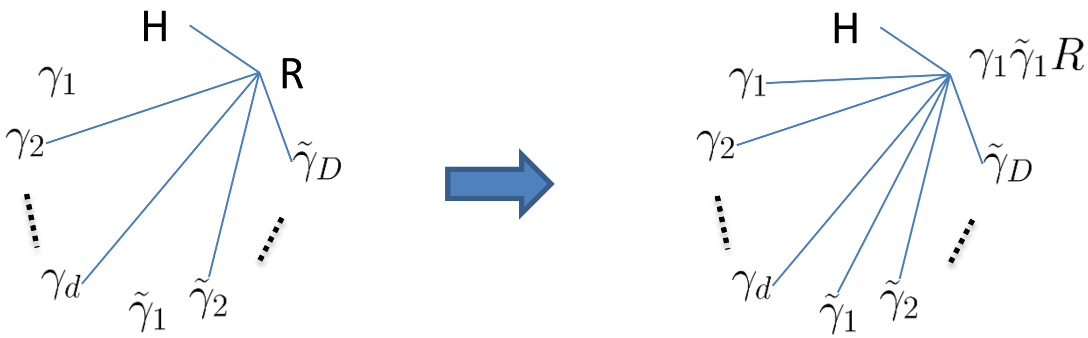
$$\{\tilde{\gamma}_a, H\} = [\tilde{\gamma}_a, J] = [\tilde{\gamma}_a, T] = \{\tilde{\gamma}_a, C\} = \{\tilde{\gamma}_a, \Gamma\} = 0$$

$$\{R, \gamma_1\} = [R, \gamma_2] = \cdots = [R, \gamma_d] = 0$$

$$\{R, \tilde{\gamma}_1\} = [R, \tilde{\gamma}_2] = \cdots = [R, \tilde{\gamma}_D] = 0$$

$$R^2 = \epsilon_R$$

Complex, class A

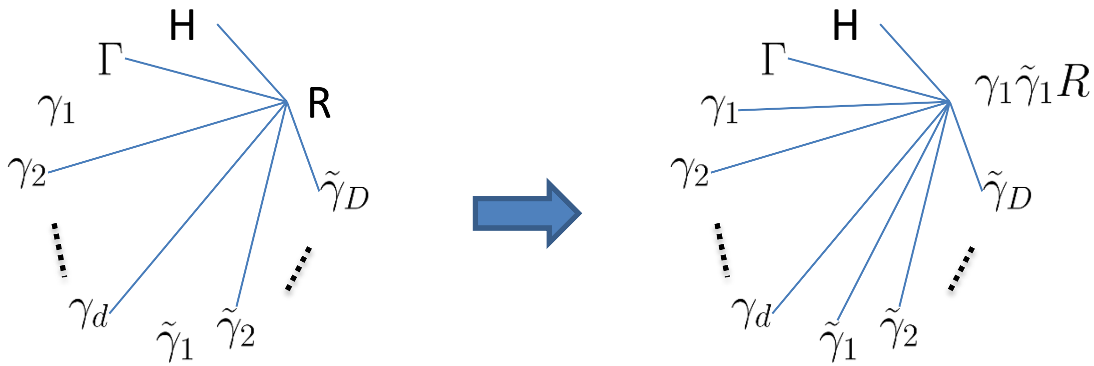


$\rightarrow decouple$

$$C_{d+D} \sim C_{d-D}$$

Complex, class AIII

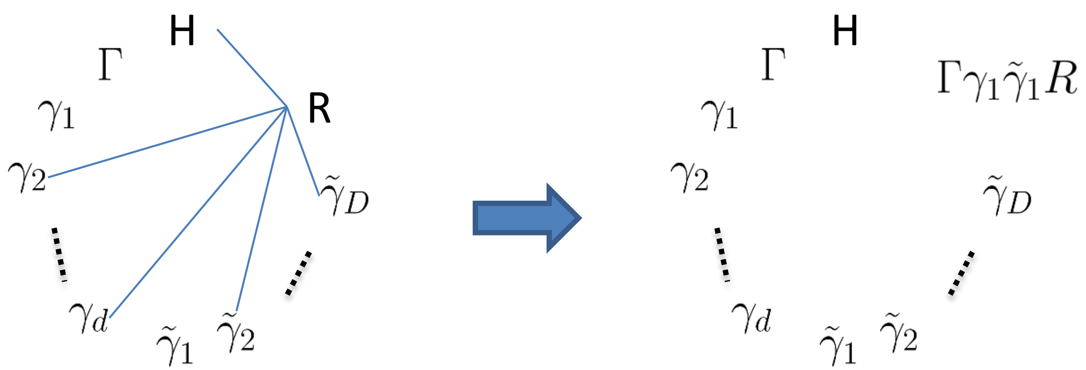
R+ : [Γ,R]=0



$\rightarrow$ *decouple*

$$C_{d+1+D} \sim C_{d+1-D}$$

R- : {Γ,R}=0

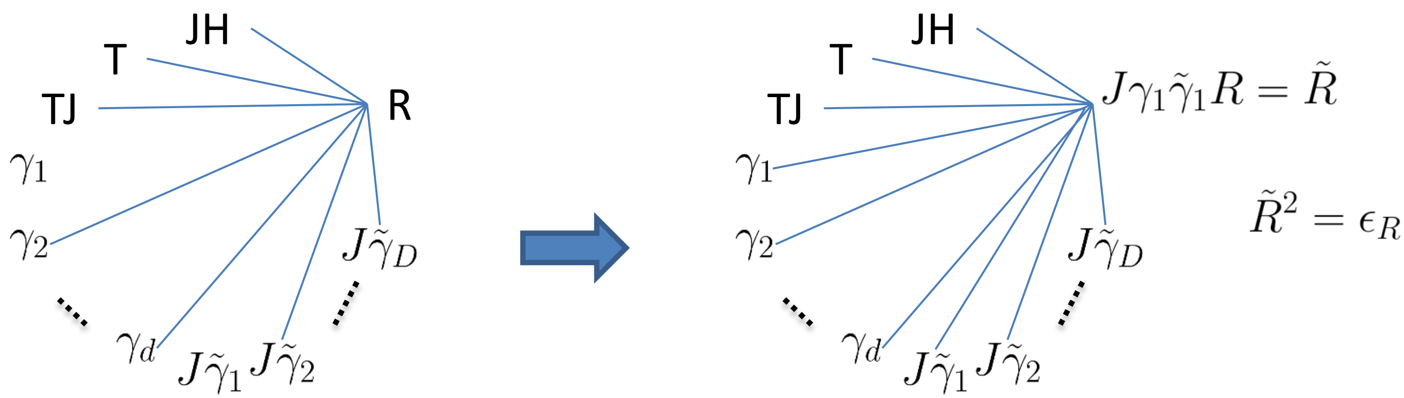


$\rightarrow +1$

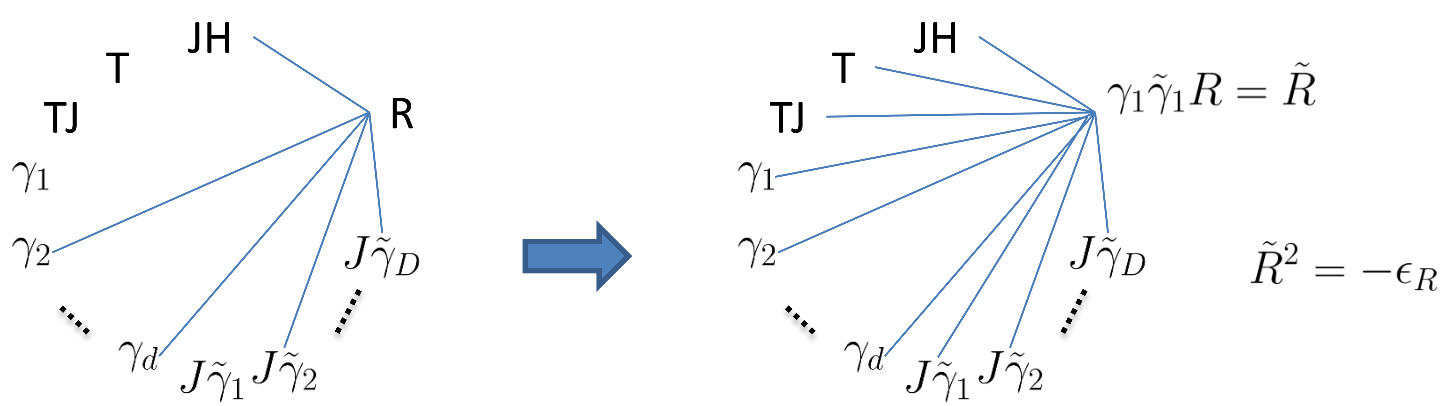
$$C_{d+1+D+1} \sim C_{d-D}$$

Real, class AI, AII

R+ : [T,R]=0

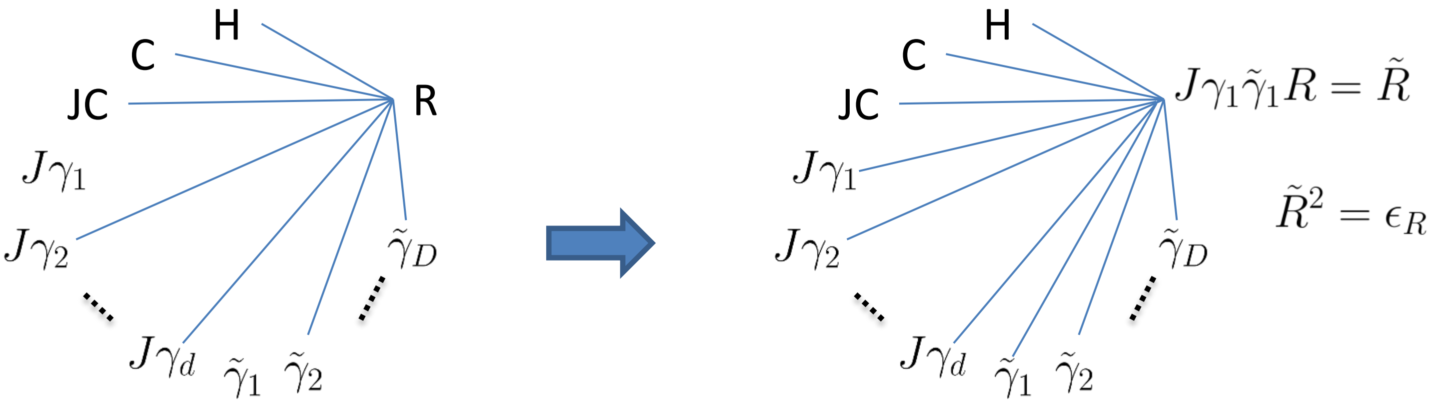


R- : {T,R}=0

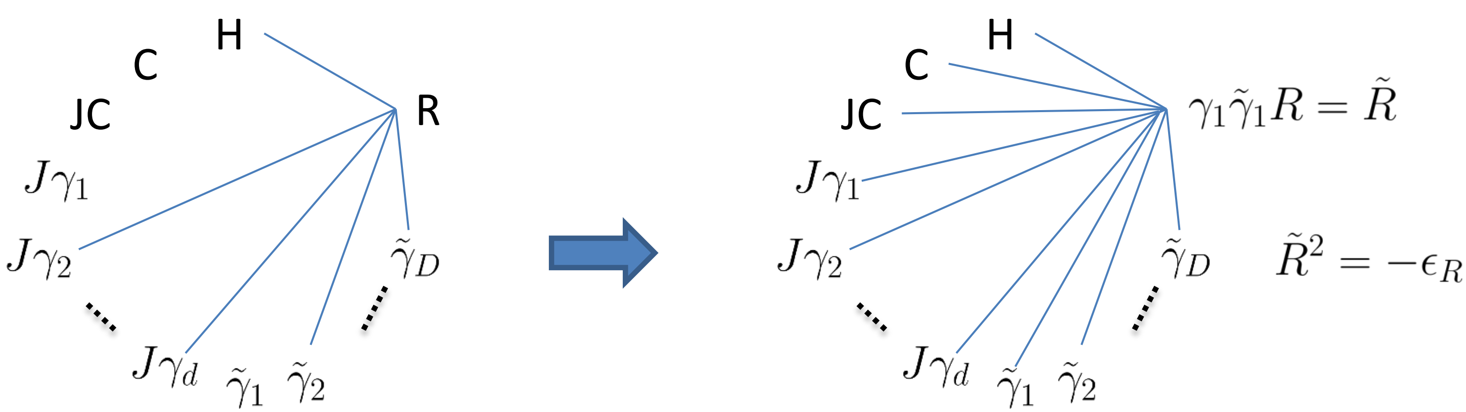


Real,   class D, C

R+ : [C,R]=0

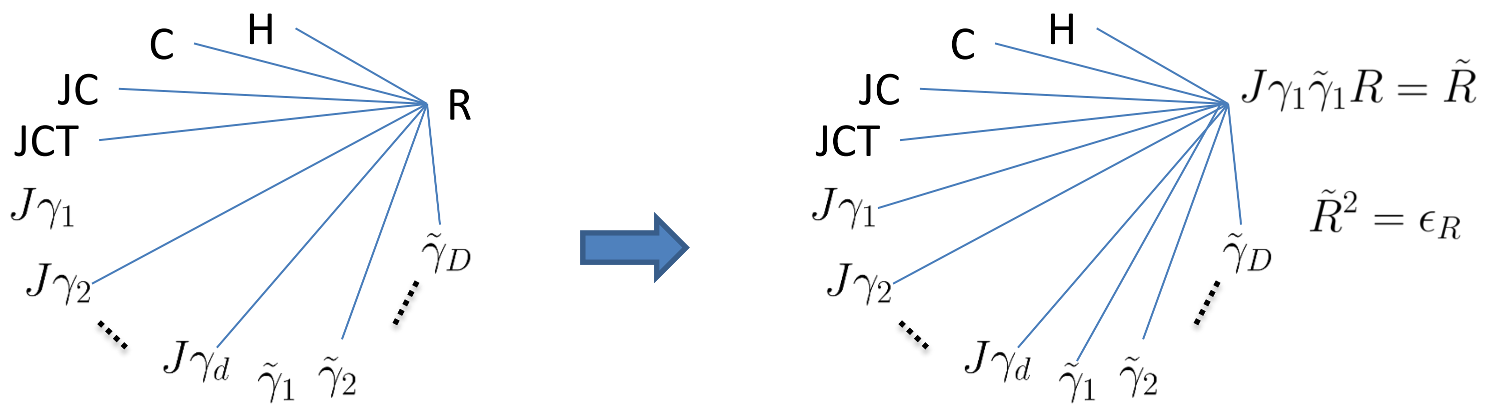


R- : {C,R}=0

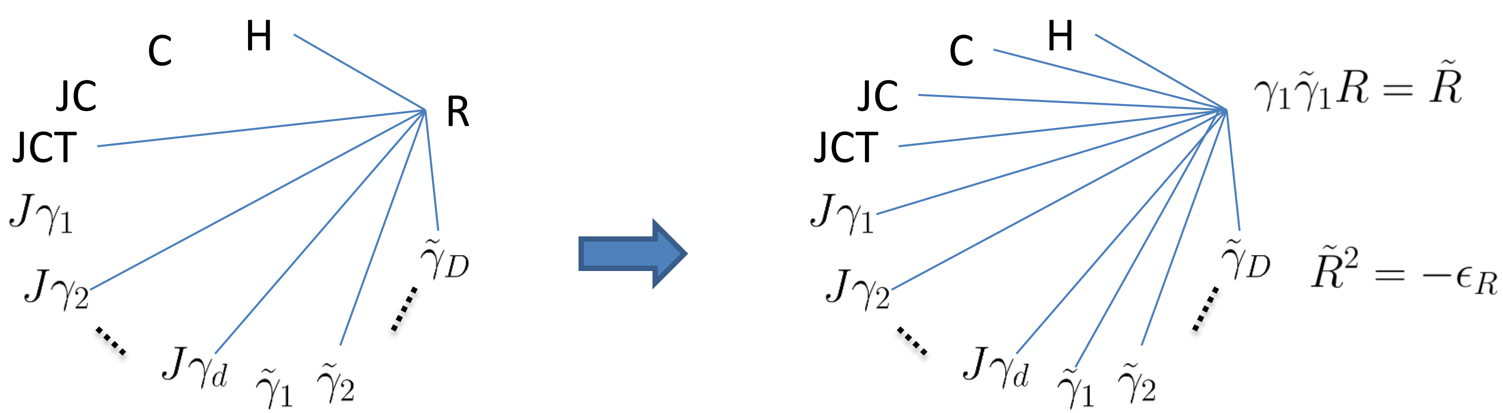


Real, class BDI, DIII, CII, CI

R++ : [T,R]=[C,R]=0

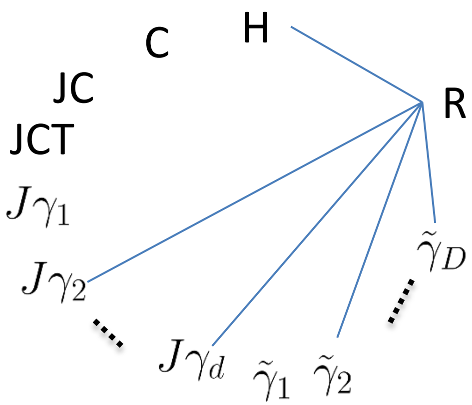


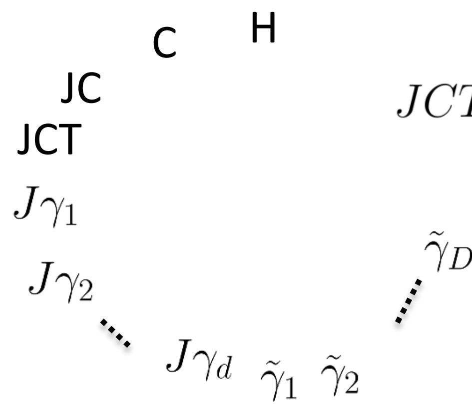
R-- : {T,R}={C,R}=0



Real, class BDI, DIII, CII, CI

R+- : [T,R]={C,R}=0

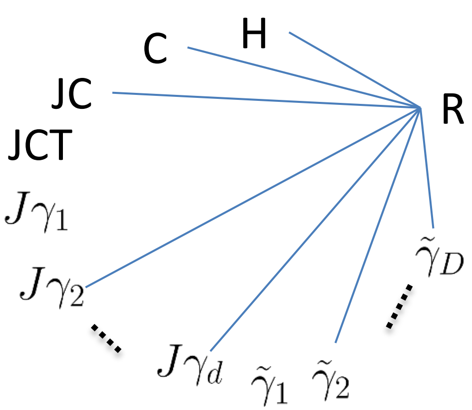


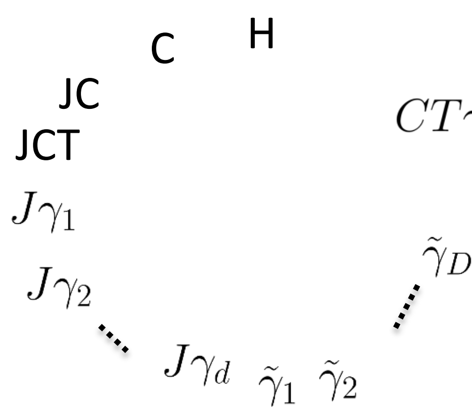


$$JCT\gamma_1\tilde{\gamma}_1 R = \tilde{R}$$

$$\tilde{R}^2 = -\epsilon_T\epsilon_C\epsilon_R$$

R-+ : {T,R}=[C,R]=0





$$CT\gamma_1\tilde{\gamma}_1 R = \tilde{R}$$

$$\tilde{R}^2 = \epsilon_T\epsilon_C\epsilon_R$$

[RDI+] Classification table for defect zero modes with unitary reflection (type I), R^2 = +

δ = d - D $R_{\eta_\Gamma},\ R_{\eta_T},\ R_{\eta_C},\ R_{\eta_T\eta_C}$

Reflection		**class**	**Cq or Rq**	**δ=0**	**δ=1**	**δ=2**	**δ=3**	**δ=4**	**δ=5**	**δ=6**	**δ=7**
R	decouple	A	C0	Z		Z		Z		Z	
R+	decouple	AIII	C1		Z		Z		Z		Z
R-	+1	AIII	C0	Z		Z		Z		Z	
R+, R++	decouple	AI	R0	Z				Z		Z2	Z2
		BDI	R1	Z2	Z				Z		Z2
		D	R2	Z2	Z2	Z				Z	
		DIII	R3		Z2	Z2	Z				Z
		AII	R4	Z		Z2	Z2	Z			
		CII	R5		Z		Z2	Z2	Z		
		C	R6			Z		Z2	Z2	Z	
		CI	R7				Z		Z2	Z2	Z
R-, R--	complex	AI, D, AII, C	C0	Z		Z		Z		Z	
		BDI, DIII, CII, CI	C1		Z		Z		Z		Z
R+-	-1	BDI	R0	Z				Z		Z2	Z2
	+1	DIII	R4	Z		Z2	Z2	Z			
	+1	CII	R6			Z		Z2	Z2	Z	
	-1	CI	R6			Z		Z2	Z2	Z	
R-+	+1	BDI	R2	Z2	Z				Z		Z2
	-1	DIII	R2	Z2	Z				Z		Z2
	-1	CII	R4		Z2	Z2	Z				Z
	+1	CI	R0				Z		Z2	Z2	Z

[RDI-] Classification table for defect zero modes with unitary reflection (type I), R^2 = -

δ = d - D $R_{\eta_\Gamma}, \; R_{\eta_T}, \; R_{\eta_C}, \; R_{\eta_T \eta_C}$

Reflection		**class**	**Cq or Rq**	**δ=0**	**δ=1**	**δ=2**	**δ=3**	**δ=4**	**δ=5**	**δ=6**	**δ=7**
R	decouple	A	C0	Z		Z		Z		Z	
R+	decouple	AIII	C1		Z		Z		Z		Z
R-	+1	AIII	C0	Z		Z		Z		Z	
R-, R--	decouple	AI	R0	Z				Z		Z2	Z2
		BDI	R1	Z2	Z				Z		Z2
		D	R2	Z2	Z2	Z				Z	
		DIII	R3		Z2	Z2	Z				Z
		AII	R4	Z		Z2	Z2	Z			
		CII	R5		Z		Z2	Z2	Z		
		C	R6			Z		Z2	Z2	Z	
		CI	R7				Z		Z2	Z2	Z
R+, R++	complex	AI, D, AII, C	C0	Z		Z		Z		Z	
		BDI, DIII, CII, CI	C1		Z		Z		Z		Z
R-+	-1	BDI	R0	Z				Z		Z2	Z2
	+1	DIII	R4	Z		Z2	Z2	Z			
	+1	CII	R6			Z		Z2	Z2	Z	
	-1	CI	R6			Z		Z2	Z2	Z	
R+-	+1	BDI	R2	Z2	Z				Z		Z2
	-1	DIII	R2	Z2	Z				Z		Z2
	-1	CII	R4		Z2	Z2	Z				Z
	+1	CI	R0				Z		Z2	Z2	Z

Unitary reflection, type (II) (D=0はboundaryの分類)

D : co-dimension of defect

$$H_{d,D} = \gamma_1 k_1 + \sum_{i=2}^{d} \gamma_i k_i + \sum_{a=1}^{D} \tilde{\gamma}_a s_a + H$$

$$\gamma_i^2 = \tilde{\gamma}_a^2 = 1$$

$$\{H, \Gamma\} = [H, J] = [H, T] = \{H, C\} = \{J, T\} = \{J, C\} = [T, C] = 0$$

$$\{\gamma_i, H\} = [\gamma_i, J] = \{\gamma_i, T\} = [\gamma_i, C] = \{\gamma_i, \Gamma\} = 0$$

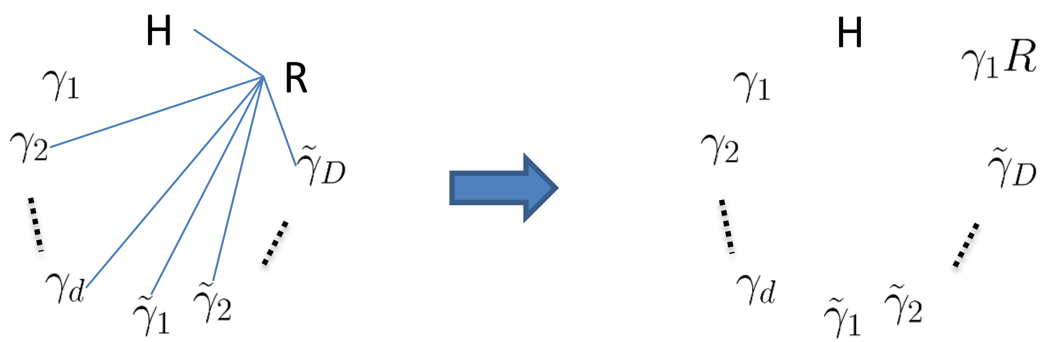
$$\{\tilde{\gamma}_a, H\} = [\tilde{\gamma}_a, J] = [\tilde{\gamma}_a, T] = \{\tilde{\gamma}_a, C\} = \{\tilde{\gamma}_a, \Gamma\} = 0$$

$$\{R, \gamma_1\} = [R, \gamma_2] = \cdots = [R, \gamma_d] = 0$$

$$[R, \tilde{\gamma}_1] = [R, \tilde{\gamma}_2] = \cdots = [R, \tilde{\gamma}_D] = 0$$

$$R^2 = \epsilon_R$$

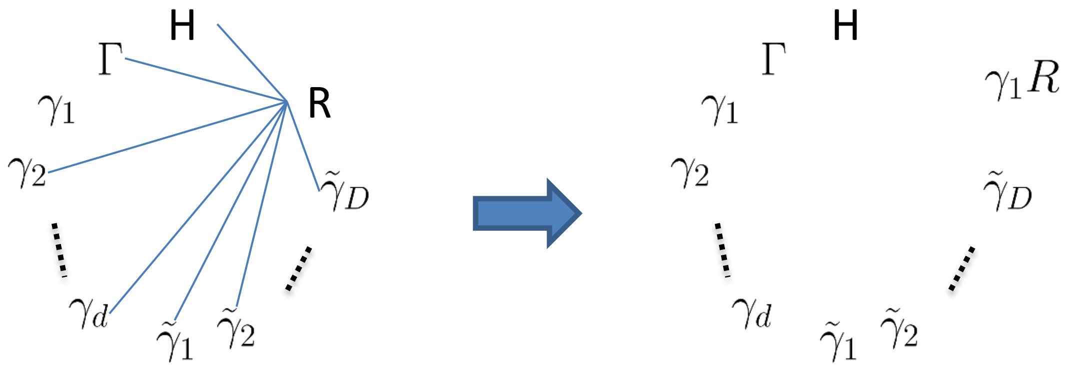
Complex, class A



$$C_{d+D+1} \sim C_{d-D+1}$$

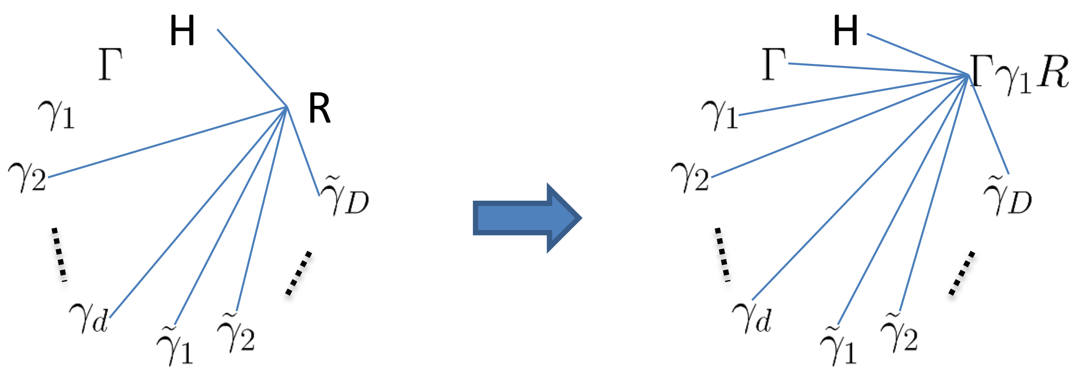
Complex, class AIII

R+ : [Γ,R]=0



$C_{d+2+D} \sim C_{d-D}$

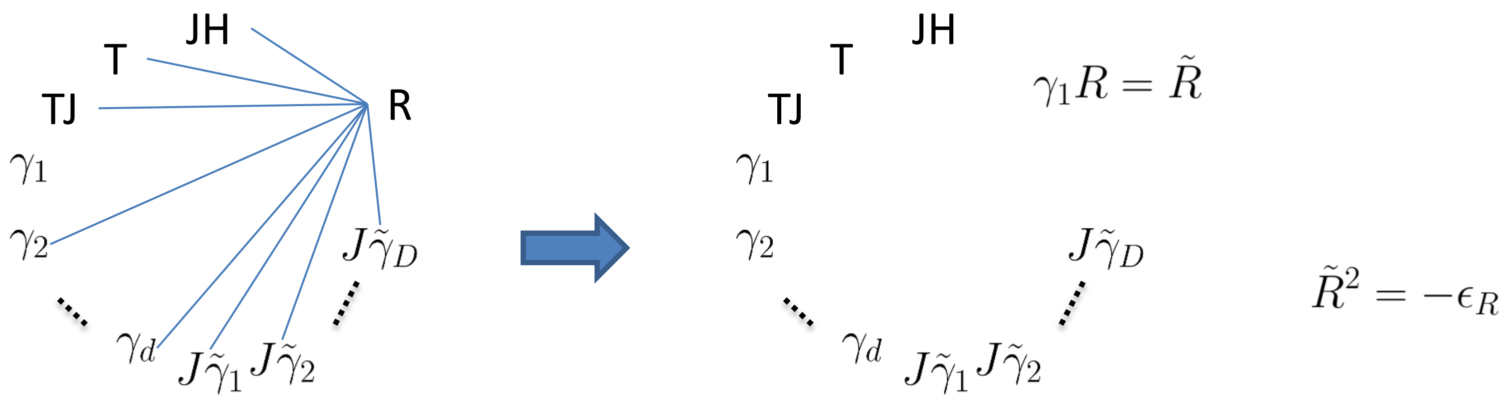
R- : {Γ,R}=0



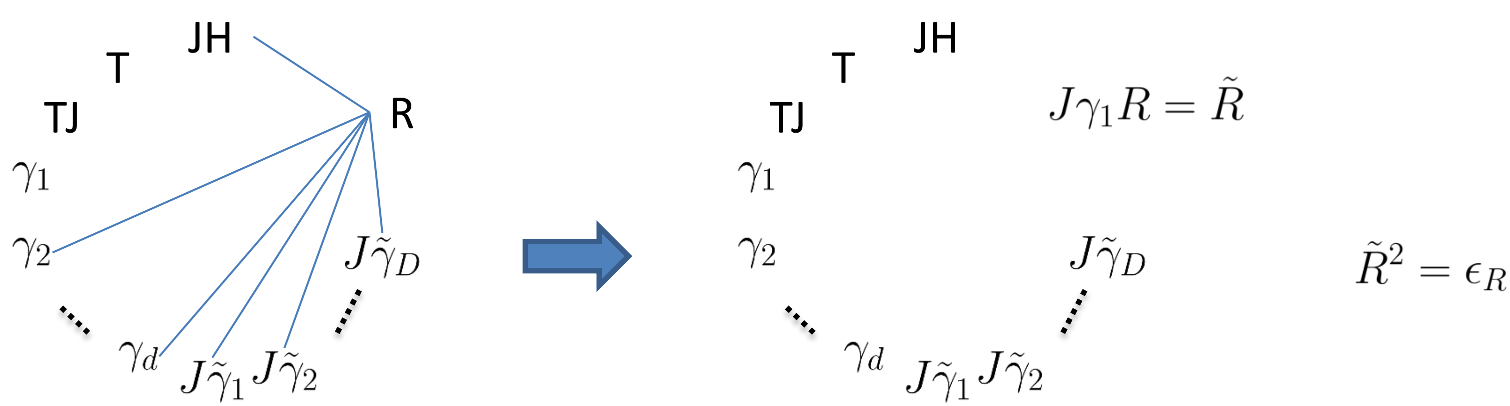
$\rightarrow decouple$

Real, class AI, AII

R+ : [T,R]=0

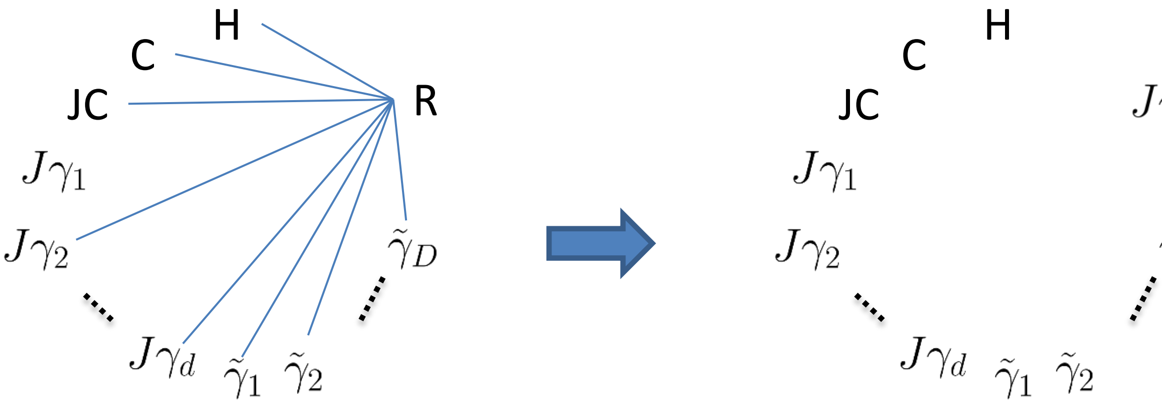


R- : {T,R}=0



Real, class D, C

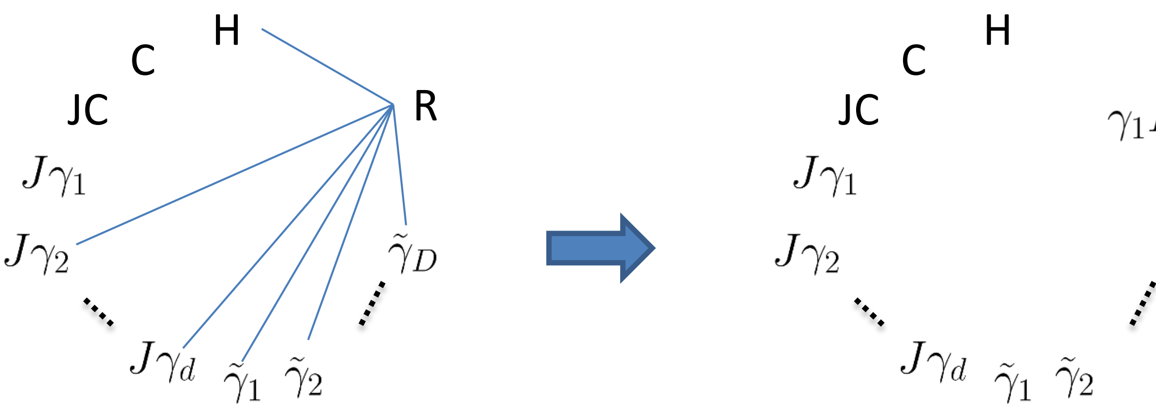
R+ : [C,R]=0



$$J\gamma_1 R = \tilde{R}$$

$$\tilde{R}^2 = \epsilon_R$$

R- : {C,R}=0

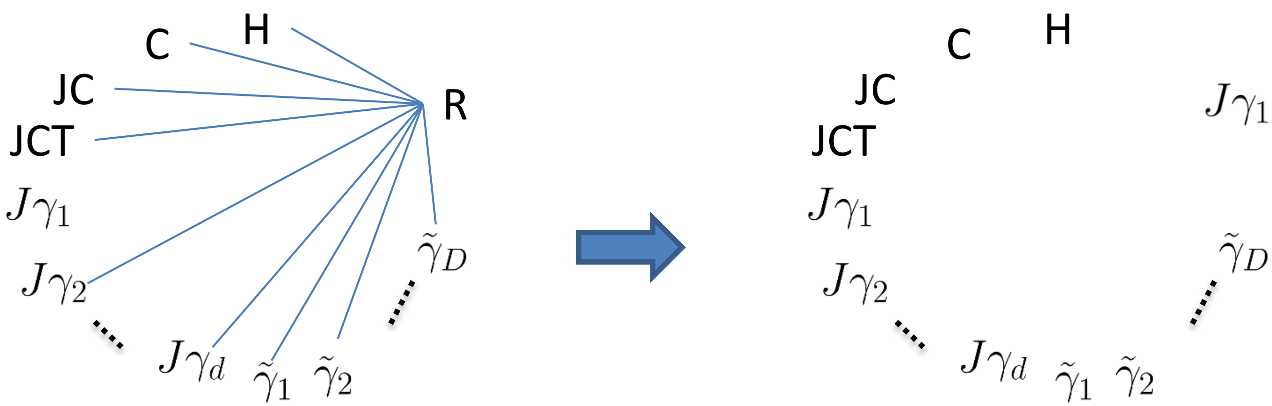


$$\gamma_1 R = \tilde{R}$$

$$\tilde{R}^2 = -\epsilon_R$$

Real,   class BDI, DIII, CII, CI

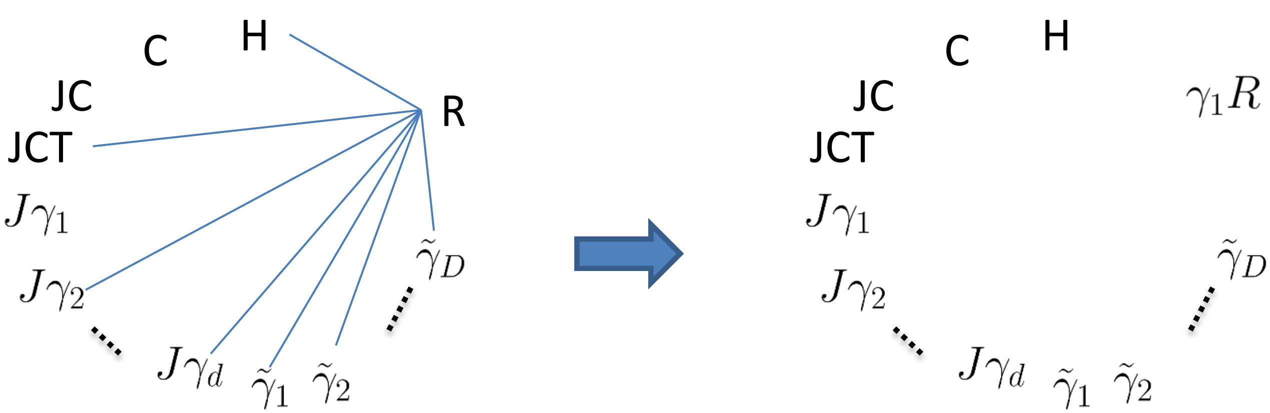
R++ : [T,R]=[C,R]=0



$$J\gamma_1 R = \tilde{R}$$

$$\tilde{R}^2 = \epsilon_R$$

R-- : {T,R}={C,R}=0

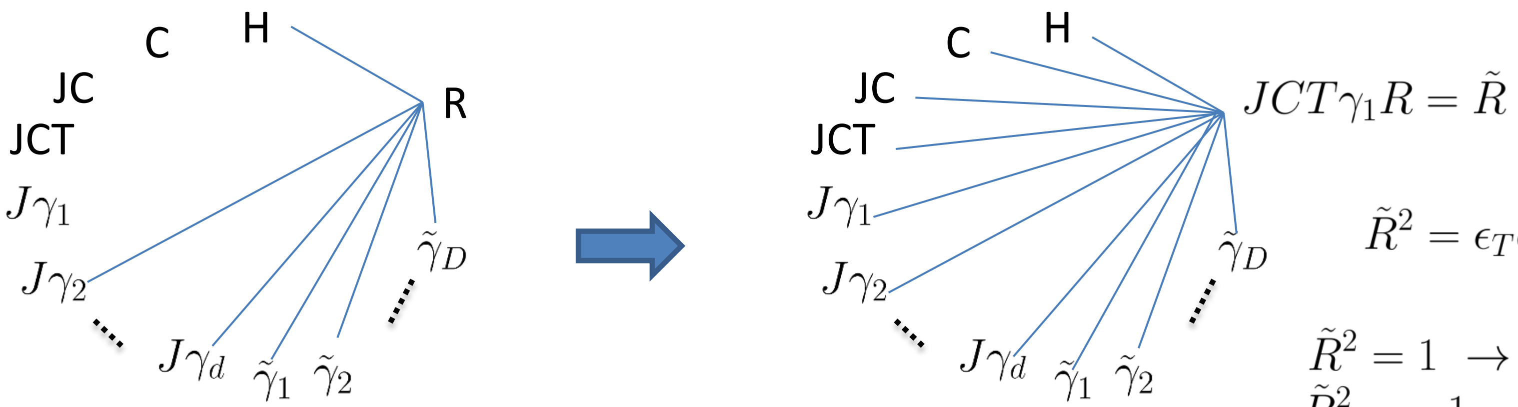


$$\gamma_1 R = \tilde{R}$$

$$\tilde{R}^2 = -\epsilon_R$$

Real, class BDI, DIII, CII, CI

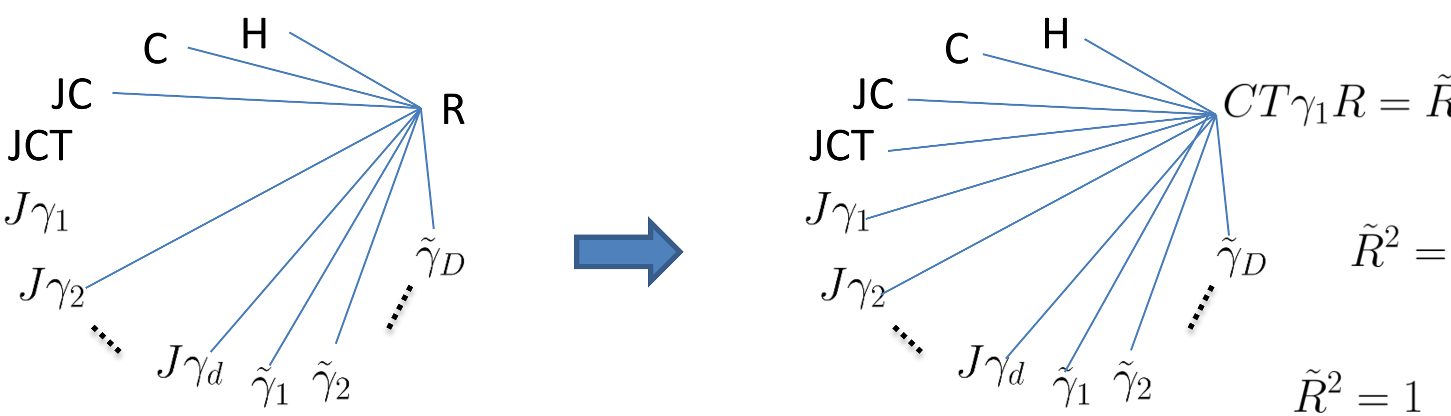
R+- : [T,R]={C,R}=0



$$\tilde{R}^2 = \epsilon_T \epsilon_C \epsilon_R$$

$$\tilde{R}^2 = 1 \ \rightarrow decouple$$
$$\tilde{R}^2 = -1 \ \rightarrow complex$$

R-+ : {T,R}=[C,R]=0



$$\tilde{R}^2 = -\epsilon_T \epsilon_C \epsilon_R$$

$$\tilde{R}^2 = 1 \ \rightarrow decouple$$
$$\tilde{R}^2 = -1 \ \rightarrow complex$$

[RDII+] Classification table for defect zero modes with unitary reflection (type II), R^2 = +

Reflection		class	Cq or Rq	δ=0	δ=1	δ=2	δ=3	δ=4	δ=5	δ=6	δ=7
R	+1	A	C1		Z		Z		Z		Z
R+	+1	AIII	C0	Z		Z		Z		Z	
R-	decouple	AIII	C1		Z		Z		Z		Z
R+, R++	+1	AI	R1	Z2	Z				Z		Z2
		BDI	R2	Z2	Z2	Z				Z	
		D	R3		Z2	Z2	Z				Z
		DIII	R4	Z		Z2	Z2	Z			
		AII	R5		Z		Z2	Z2	Z		
		CII	R6			Z		Z2	Z2	Z	
		C	R7				Z		Z2	Z2	Z
		CI	R0	Z				Z		Z2	Z2
R-, R--	-1	AI	R7				Z		Z2	Z2	Z
		BDI	R0	Z				Z		Z2	Z2
		D	R1	Z2	Z				Z		Z2
		DIII	R2	Z2	Z2	Z				Z	
		AII	R3		Z2	Z2	Z				Z
		CII	R4	Z		Z2	Z2	Z			
		C	R5		Z		Z2	Z2	Z		
		CI	R6			Z		Z2	Z2	Z	
R+-	decouple	BDI	R1	Z2	Z				Z		Z2
R-+		DIII	R3		Z2	Z2	Z				Z
R+-		CII	R5		Z		Z2	Z2	Z		
R-+		CI	R7				Z		Z2	Z2	Z
R-+	complex	BDI, CII	C1		Z		Z		Z		Z
R+-		DIII, CI	C1		Z		Z		Z		Z

[RDII-] Classification table for defect zero modes with unitary reflection (type II), R^2 = -

Reflection		class	Cq or Rq	δ=0	δ=1	δ=2	δ=3	δ=4	δ=5	δ=6	δ=7
R	+1	A	C1		Z		Z		Z		Z
R+	+1	AIII	C0	Z		Z		Z		Z	
R-	decouple	AIII	C1		Z		Z		Z		Z
R-, R--	+1	AI	R1	Z2	Z				Z		Z2
		BDI	R2	Z2	Z2	Z				Z	
		D	R3		Z2	Z2	Z				Z
		DIII	R4	Z		Z2	Z2	Z			
		AII	R5		Z		Z2	Z2	Z		
		CII	R6			Z		Z2	Z2	Z	
		C	R7				Z		Z2	Z2	Z
		CI	R0	Z				Z		Z2	Z2
R+, R++	-1	AI	R7				Z		Z2	Z2	Z
		BDI	R0	Z				Z		Z2	Z2
		D	R1	Z2	Z				Z		Z2
		DIII	R2	Z2	Z2	Z				Z	
		AII	R3		Z2	Z2	Z				Z
		CII	R4	Z		Z2	Z2	Z			
		C	R5		Z		Z2	Z2	Z		
		CI	R6			Z		Z2	Z2	Z	
R-+	decouple	BDI	R1	Z2	Z				Z		Z2
R+-		DIII	R3		Z2	Z2	Z				Z
R-+		CII	R5		Z		Z2	Z2	Z		
R+-		CI	R7				Z		Z2	Z2	Z
R+-	complex	BDI, CII	C1		Z		Z		Z		Z
R-+		DIII, CI	C1		Z		Z		Z		Z

Anti-unitary reflection, type (I) (D≧1)

D : co-dimension of defect

$$H_{d,D} = \gamma_1 k_1 + \sum_{i=2}^{d} \gamma_i k_i + \tilde{\gamma}_1 s_1 + \sum_{a=2}^{D} \tilde{\gamma}_a s_a + H$$

$$\gamma_i^2 = \tilde{\gamma}_a^2 = 1$$

$$\{H, \Gamma\} = [H, J] = [H, T] = \{H, C\} = \{J, T\} = \{J, C\} = [T, C] = 0$$

$$\{\gamma_i, H\} = [\gamma_i, J] = \{\gamma_i, T\} = [\gamma_i, C] = \{\gamma_i, \Gamma\} = 0$$

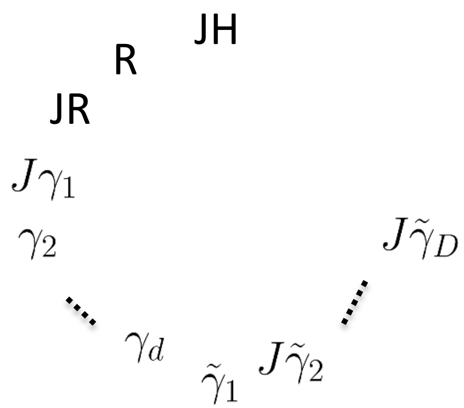
$$\{\tilde{\gamma}_a, H\} = [\tilde{\gamma}_a, J] = [\tilde{\gamma}_a, T] = \{\tilde{\gamma}_a, C\} = \{\tilde{\gamma}_a, \Gamma\} = 0$$

$$[R, \gamma_1] = \{R, \gamma_2\} = \cdots = \{R, \gamma_d\} = 0$$

$$\{R, \tilde{\gamma}_1\} = [R, \tilde{\gamma}_2] = \cdots = [R, \tilde{\gamma}_D] = 0$$

$$R^2 = \epsilon_R$$

Complex, class A → real

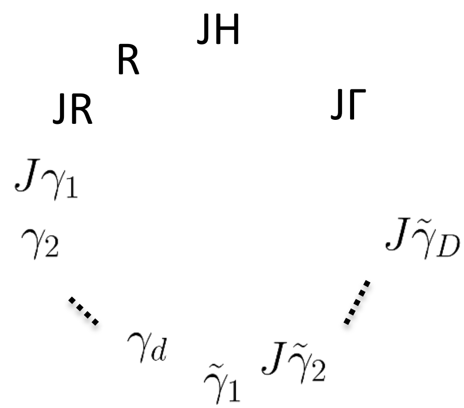


$$\epsilon_R = + \ : \ Cl_{D,d+2} \to Cl_{D+1,d+2} \ : \ R_{-d+D}$$

$$\epsilon_R = - \ : \ Cl_{D+2,d} \to Cl_{D+3,d} \ : \ R_{4-d+D}$$

Complex, class AIII → real

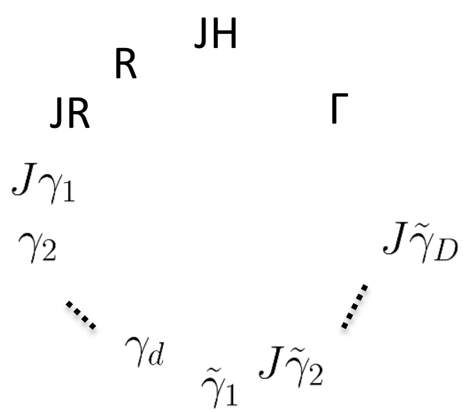
R+ : [Γ,R]=0



$$\epsilon_R = + \;:\; Cl_{D+1,d+2} \to Cl_{D+2,d+2} \;:\; R_{1-d+D}$$

$$\epsilon_R = - \;:\; Cl_{D+3,d} \to Cl_{D+4,d} \;:\; R_{5-d+D}$$

R- : {Γ,R}=0

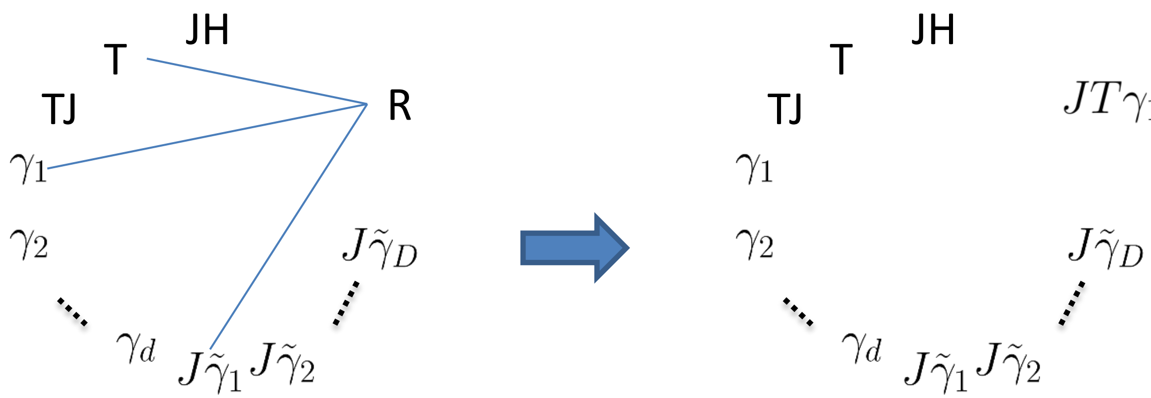


$$\epsilon_R = + \;:\; Cl_{D,d+3} \to Cl_{D+1,d+3} \;:\; R_{7-d+D}$$

$$\epsilon_R = - \;:\; Cl_{D+2,d+1} \to Cl_{D+3,d+1} \;:\; R_{3-d+D}$$

Real, class AI, AII

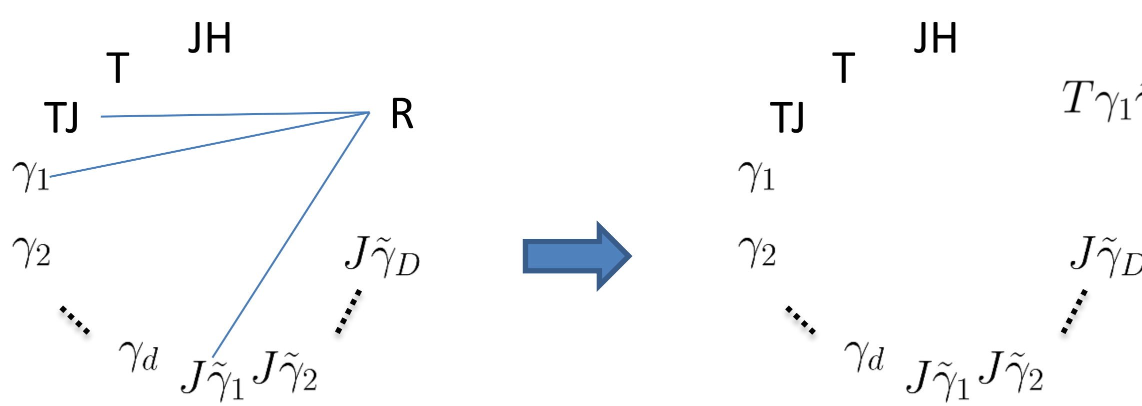
R+ : [T,R]=0



$JT\gamma_1\tilde{\gamma}_1 R = \tilde{R}$

$\tilde{R}^2 = \epsilon_T \epsilon_R$

R- : {T,R}=0

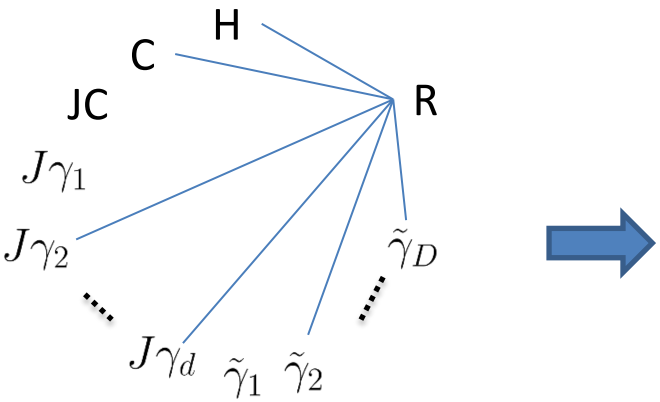


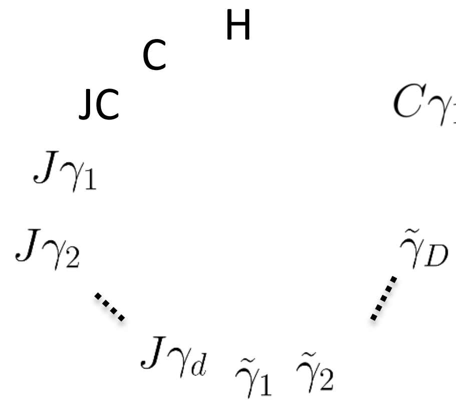
$T\gamma_1\tilde{\gamma}_1 R = \tilde{R}$

$\tilde{R}^2 = \epsilon_T \epsilon_R$

Real, class D, C

R+ : [C,R]=0

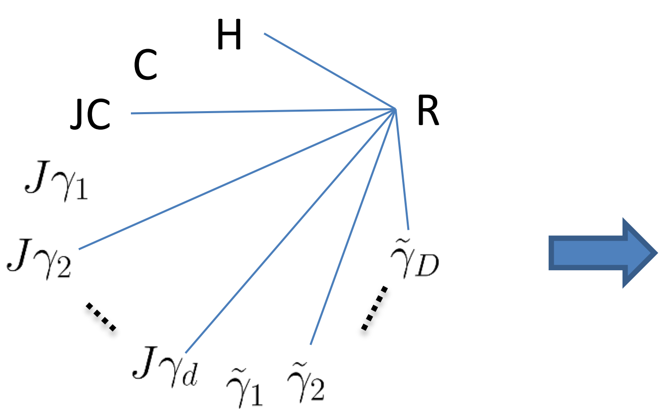


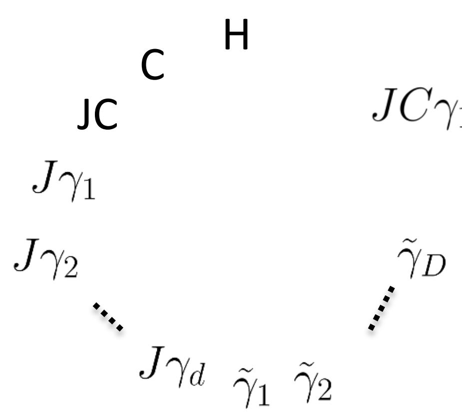


$$C\gamma_1\tilde{\gamma}_1 R = \tilde{R}$$

$$\tilde{R}^2 = -\epsilon_C\epsilon_R$$

R- : {C,R}=0



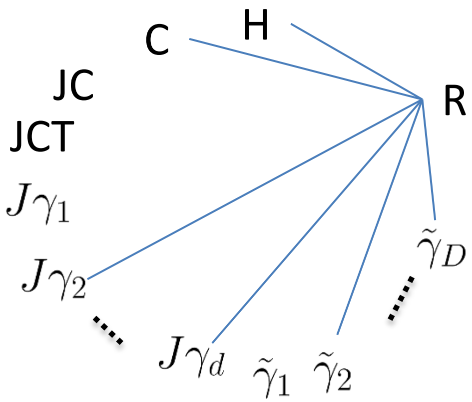


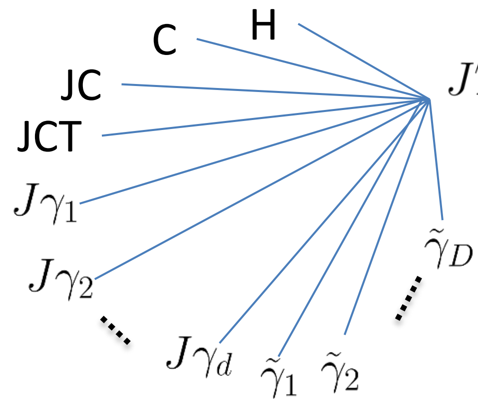
$$JC\gamma_1\tilde{\gamma}_1 R = \tilde{R}$$

$$\tilde{R}^2 = -\epsilon_C\epsilon_R$$

Real, class BDI, DIII, CII, CI

R++ : [T,R]=[C,R]=0



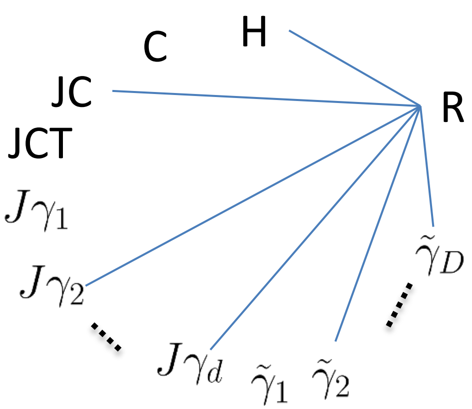


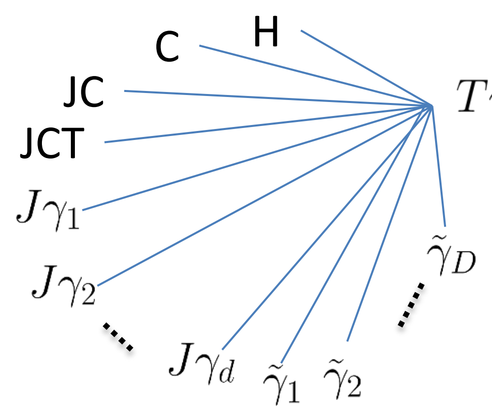
$$JT\gamma_1\tilde{\gamma}_1 R = \tilde{R}$$

$$\tilde{R}^2 = \epsilon_T \epsilon_R$$

$$\tilde{R}^2 = 1 \rightarrow decouple$$
$$\tilde{R}^2 = -1 \rightarrow complex$$

R-- : {T,R}={C,R}=0





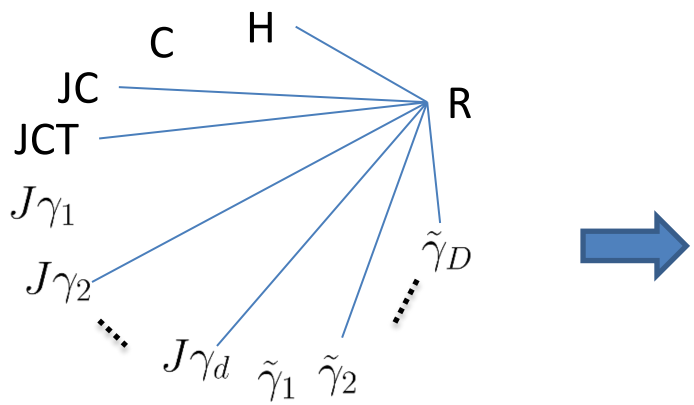
$$T\gamma_1\tilde{\gamma}_1 R = \tilde{R}$$

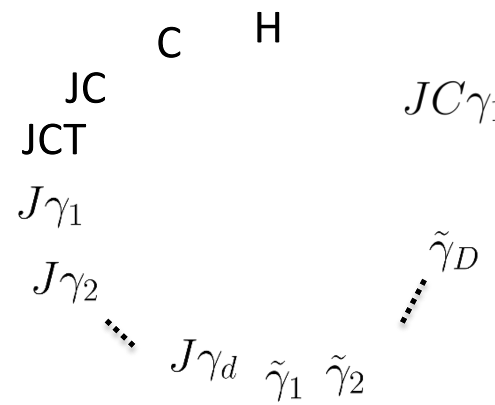
$$\tilde{R}^2 = \epsilon_T \epsilon_R$$

$$\tilde{R}^2 = 1 \rightarrow decouple$$
$$\tilde{R}^2 = -1 \rightarrow complex$$

Real,   class BDI, DIII, CII, CI

R+- : [T,R]={C,R}=0

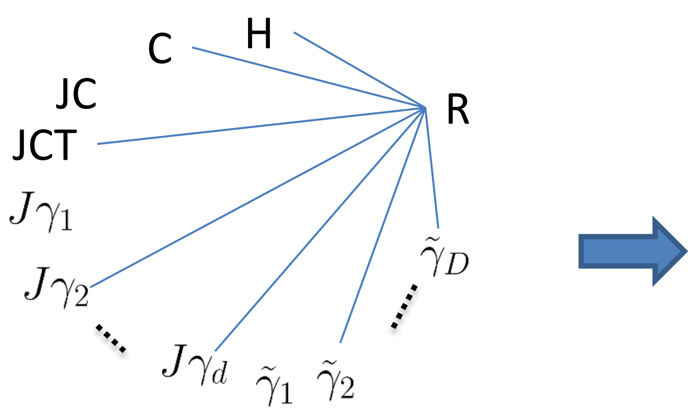


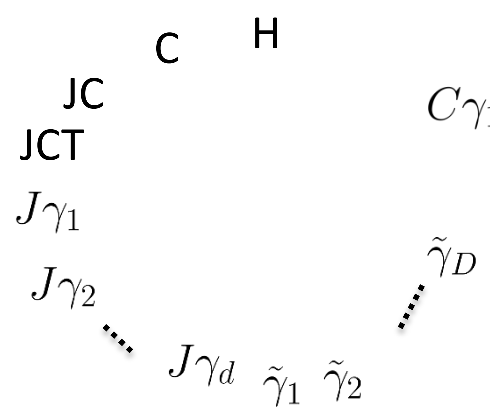


$$JC\gamma_1\tilde{\gamma}_1 R = \tilde{R}$$

$$\tilde{R}^2 = -\epsilon_C\epsilon_R$$

R-+ : {T,R}=[C,R]=0





$$C\gamma_1\tilde{\gamma}_1 R = \tilde{R}$$

$$\tilde{R}^2 = -\epsilon_C\epsilon_R$$

[aRDI+] Classification table for defect zero modes with anti-unitary reflection (type I), $R^2 = 1$

class	Reflection		Cq or Rq	δ=0	δ=1	δ=2	δ=3	δ=4	δ=5	δ=6	δ=7
A	R	real	R0	Z				Z		Z2	Z2
AIII	R+	Real	R1	Z2	Z				Z		Z2
	R-	real	R7				Z		Z2	Z2	Z
AI	R	-1	R7				Z		Z2	Z2	Z
AII	R	+1	R5		Z		Z2	Z2	Z		
D	R	-1	R1	Z2	Z				Z		Z2
C	R	+1	R7				Z		Z2	Z2	Z
BDI	R++, R--	Decouple	R1	Z2	Z				Z		Z2
	R+-, R-+	-1	R0	Z				Z		Z2	Z2
DIII	R++, R--	Complex	C1		Z		Z		Z		Z
	R+-, R-+	-1	R2	Z2	Z2	Z				Z	
CII	R++, R--	Complex	C1		Z		Z		Z		Z
	R+-, R-+	+1	R6			Z		Z2	Z2	Z	
CI	R++, R--	Decouple	R7				Z		Z2	Z2	Z
	R+-, R-+	+1	R0	Z				Z		Z2	Z2

[aRDI-] Classification table for defect zero modes with anti-unitary reflection (type I), R^2 = -1

class	Reflection		Cq or Rq	δ=0	δ=1	δ=2	δ=3	δ=4	δ=5	δ=6	δ=7
A	R	real	R4	Z		Z2	Z2	Z			
AIII	R+	Real	R5		Z		Z2	Z2	Z		
	R-	real	R3		Z2	Z2	Z				Z
AI	R	+1	R1	Z2	Z				Z		Z2
AII	R	-1	R3		Z2	Z2	Z				Z
D	R	+1	R3		Z2	Z2	Z				Z
C	R	-1	R5		Z		Z2	Z2	Z		
BDI	R++, R--	complex	C1		Z		Z		Z		Z
	R+-, R-+	+1	R2	Z2	Z2	Z				Z	
DIII	R++, R--	decouple	R3		Z2	Z2	Z				Z
	R+-, R-+	+1	R4	Z		Z2	Z2	Z			
CII	R++, R--	decouple	R5		Z		Z2	Z2	Z		
	R+-, R-+	-1	R4	Z		Z2	Z2	Z			
CI	R++, R--	complex	C1		Z		Z		Z		Z
	R+-, R-+	-1	R6			Z		Z2	Z2	Z	

Anti-unitary reflection, type (II) (D=0はboundaryの分類)

D : co-dimension of defect

$$H_{d,D} = \gamma_1 k_1 + \sum_{i=2}^{d} \gamma_i k_i + \sum_{a=1}^{D} \tilde{\gamma}_a s_a + H$$

$$\gamma_i^2 = \tilde{\gamma}_a^2 = 1$$

$$\{H, \Gamma\} = [H, J] = [H, T] = \{H, C\} = \{J, T\} = \{J, C\} = [T, C] = 0$$

$$\{\gamma_i, H\} = [\gamma_i, J] = \{\gamma_i, T\} = [\gamma_i, C] = \{\gamma_i, \Gamma\} = 0$$

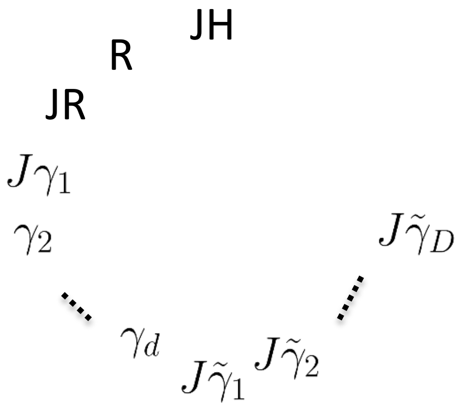
$$\{\tilde{\gamma}_a, H\} = [\tilde{\gamma}_a, J] = [\tilde{\gamma}_a, T] = \{\tilde{\gamma}_a, C\} = \{\tilde{\gamma}_a, \Gamma\} = 0$$

$$[R, \gamma_1] = \{R, \gamma_2\} = \cdots = \{R, \gamma_d\} = 0$$

$$[R, \tilde{\gamma}_1] = [R, \tilde{\gamma}_2] = \cdots = [R, \tilde{\gamma}_D] = 0$$

$$R^2 = \epsilon_R$$

Complex, class A → real

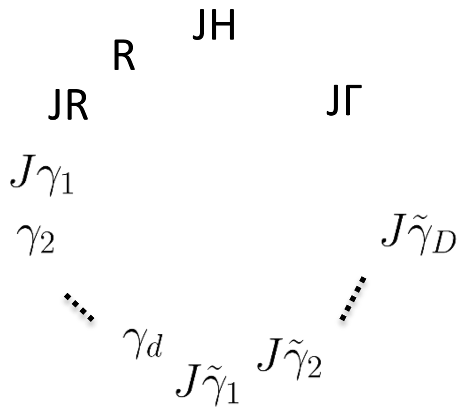


$$\epsilon_R = + \ : \ Cl_{D+1,d+1} \to Cl_{D+2,d+1} \ : \ R_{2-d+D}$$

$$\epsilon_R = - \ : \ Cl_{D+3,d-1} \to Cl_{D+4,d-1} \ : \ R_{6-d+D}$$

Complex, class AIII → real

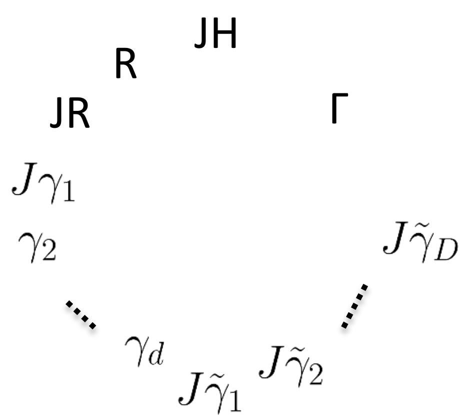
R+ : [Γ,R]=0



$$\epsilon_R = + \;:\; Cl_{D+2,d+1} \to Cl_{D+3,d+1} \;:\; R_{3-d+D}$$

$$\epsilon_R = - \;:\; Cl_{D+4,d-1} \to Cl_{D+5,d-1} \;:\; R_{7-d+D}$$

R- : {Γ,R}=0



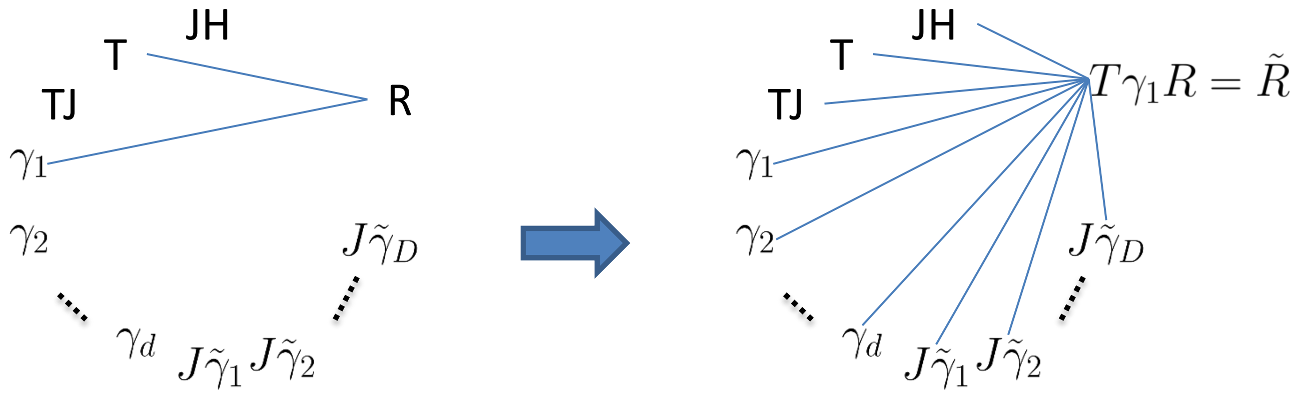
$$\epsilon_R = + \;:\; Cl_{D+1,d+2} \to Cl_{D+2,d+2} \;:\; R_{1-d+D}$$

$$\epsilon_R = - \;:\; Cl_{D+3,d} \to Cl_{D+4,d} \;:\; R_{5-d+D}$$

Real,　class AI, AII

R+ : [T,R]=0

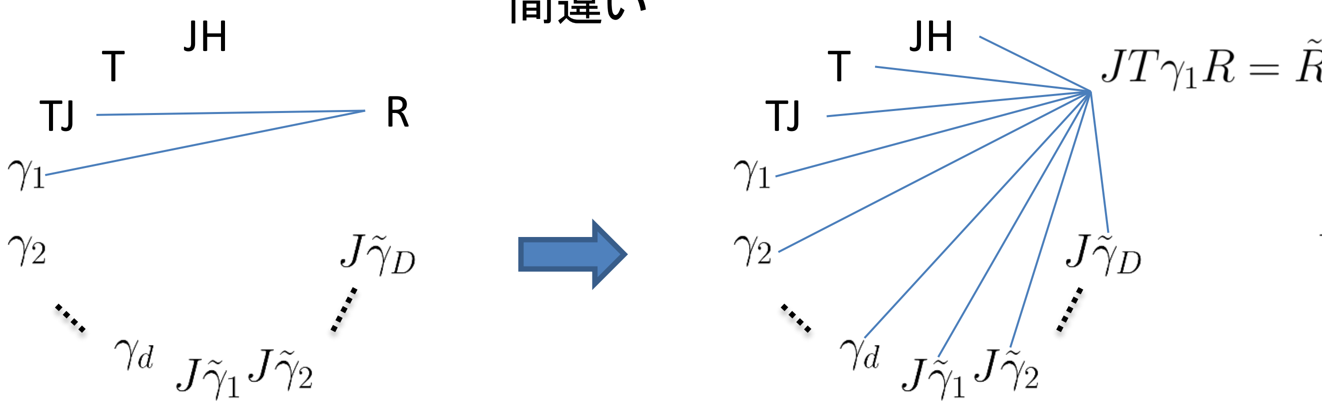
間違い



$$\tilde{R}^2 = -\epsilon_T \epsilon_R$$

R- : {T,R}=0

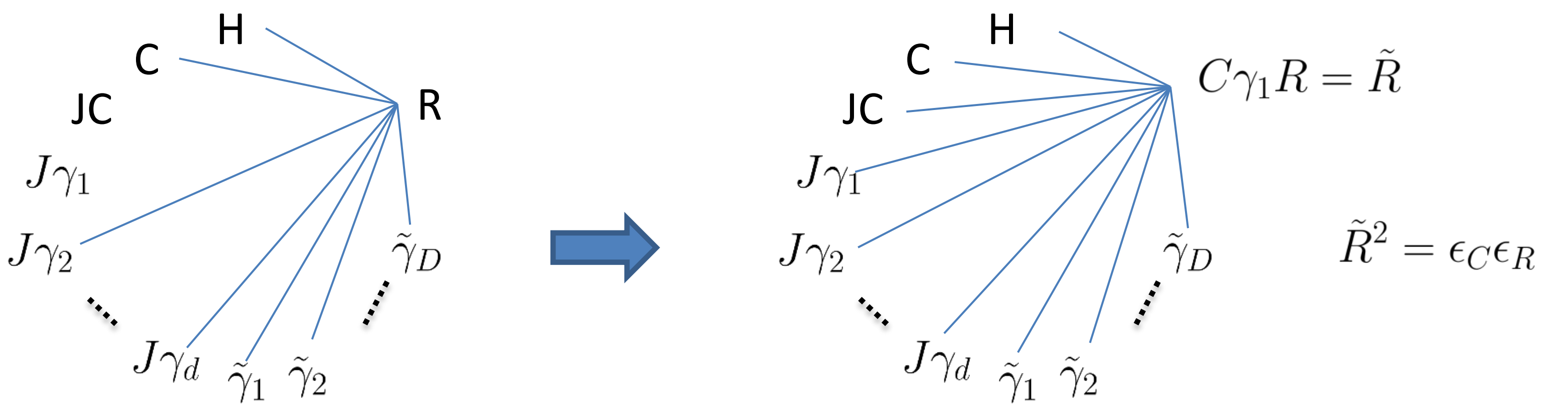
間違い



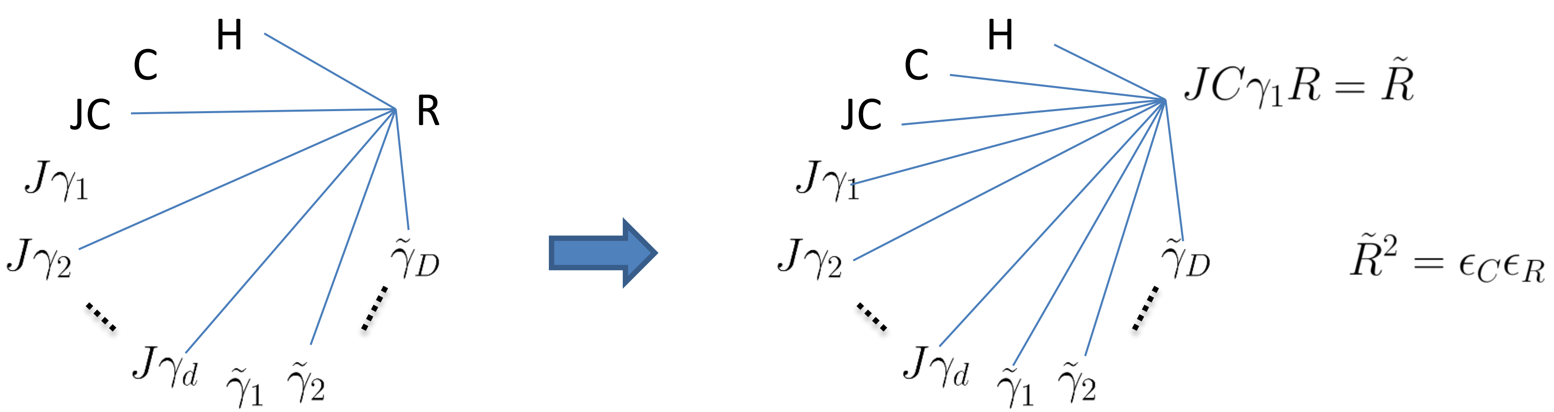
$$\tilde{R}^2 = -\epsilon_T \epsilon_R$$

Real, class D, C

R+ : [C,R]=0

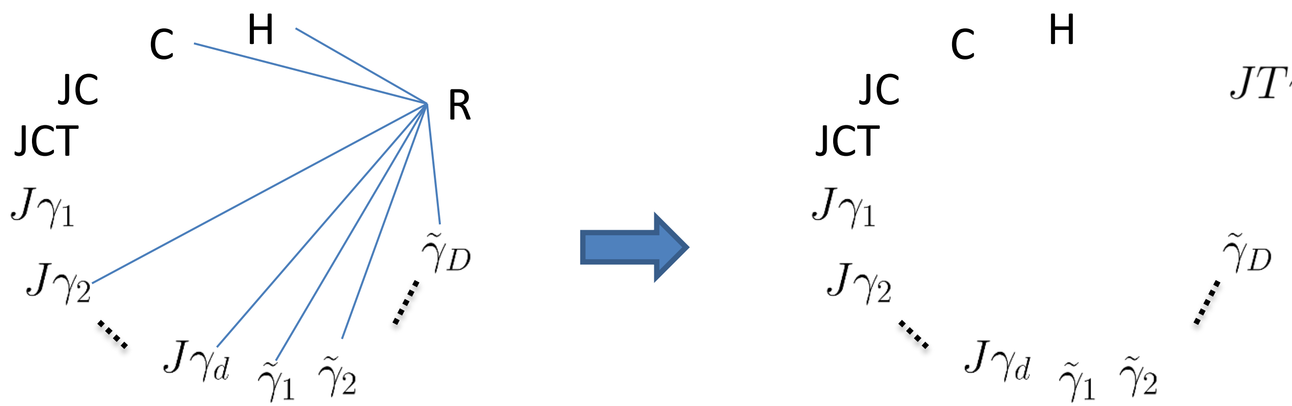


R- : {C,R}=0



Real, class BDI, DIII, CII, CI

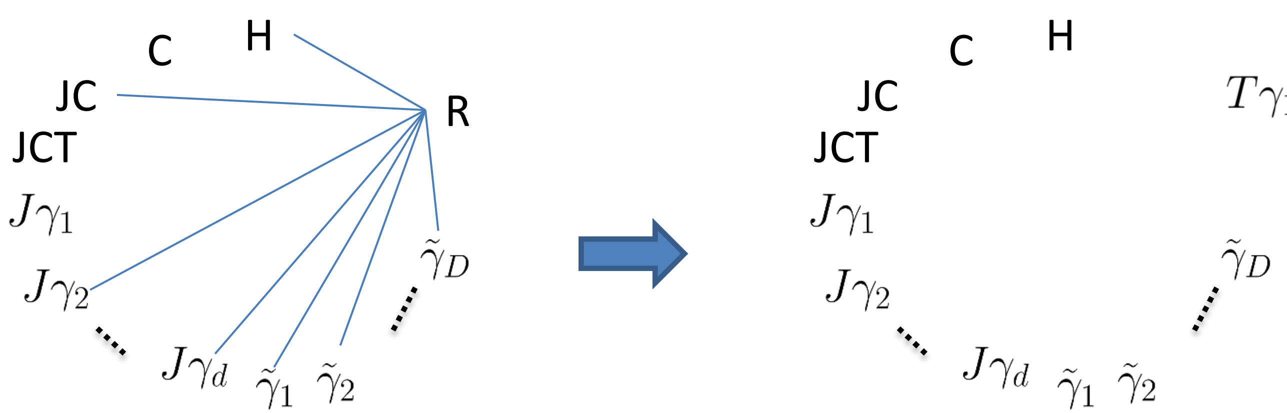
R++ : [T,R]=[C,R]=0



$$JT\gamma_1 R = \tilde{R}$$

$$\tilde{R}^2 = \epsilon_T \epsilon_R$$

R-- : {T,R}={C,R}=0

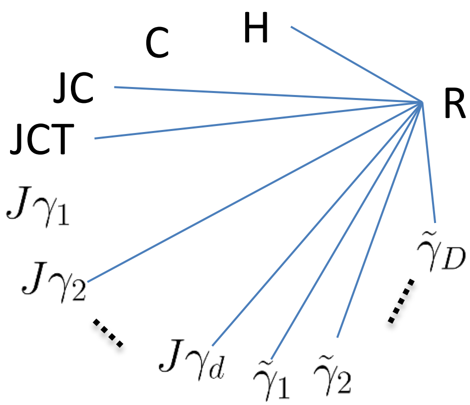


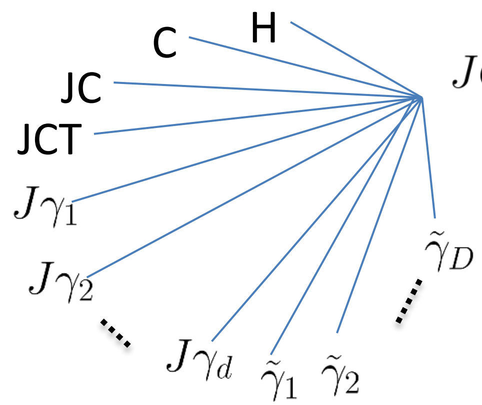
$$T\gamma_1 R = \tilde{R}$$

$$\tilde{R}^2 = \epsilon_T \epsilon_R$$

Real,   class BDI, DIII, CII, CI

R+- : [T,R]={C,R}=0

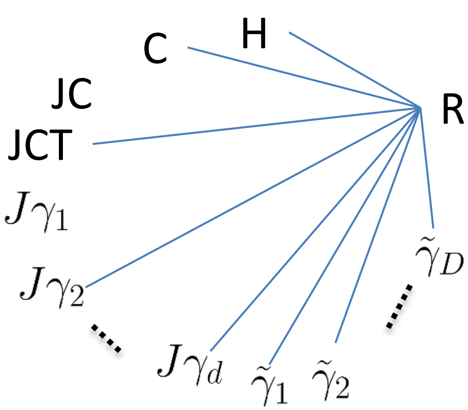


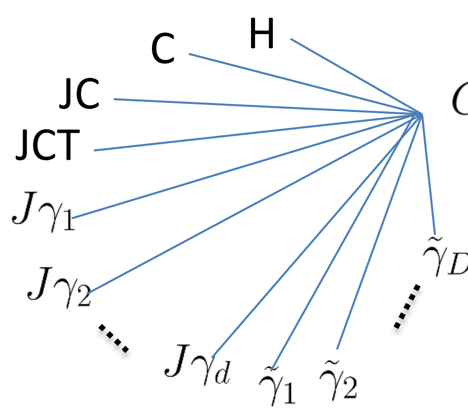


$$JC\gamma_1 R = \tilde{R}$$

$$\tilde{R}^2 = \epsilon_C \epsilon_R$$

R-+ : {T,R}=[C,R]=0





$$C\gamma_1 R = \tilde{R}$$

$$\tilde{R}^2 = \epsilon_C \epsilon_R$$

[aRDII+] Classification table for defect zero modes with anti-unitary reflection (type II), $R^2 = +$ (特にD=0はboundaryの分類)

class	Reflection		Cq or Rq	δ=0	δ=1	δ=2	δ=3	δ=4	δ=5	δ=6	δ=7
A	R	real	R2	Z2	Z2	Z				Z	
AIII	R+	Real	R3		Z2	Z2	Z				Z
	R-	real	R1	Z2	Z				Z		Z2
AI	R	Complex	C0	Z		Z		Z		Z	
AII	R	decouple	R4	Z		Z2	Z2	Z			
D	R	Decouple	R2	Z2	Z2	Z				Z	
C	R	complex	C0	Z		Z		Z		Z	
BDI	R++, R--	+1	R2	Z2	Z2	Z				Z	
	R+-, R-+	Decouple	R1	Z2	Z				Z		Z2
DIII	R++, R--	-1	R2	Z2	Z2	Z				Z	
	R+-, R-+	Decouple	R3		Z2	Z2	Z				Z
CII	R++, R--	-1	R4	Z		Z2	Z2	Z			
	R+-, R-+	Complex	C1		Z		Z		Z		Z
CI	R++, R--	+1	R0				Z		Z2	Z2	Z
	R+-, R-+	complex	C1		Z		Z		Z		Z

[aRDII-] Classification table for defect zero modes with anti-unitary reflection (type II), R^2 = - (特にD=0はboundaryの分類)

class	Reflection		Cq or Rq	δ=0	δ=1	δ=2	δ=3	δ=4	δ=5	δ=6	δ=7
A	R	real	R6			Z		Z2	Z2	Z	
AIII	R+	Real	R7				Z		Z2	Z2	Z
	R-	real	R5		Z		Z2	Z2	Z		
AI	R	decouple	R0	Z				Z		Z2	Z2
AII	R	complex	C0	Z		Z		Z		Z	
D	R	complex	C0	Z		Z		Z		Z	
C	R	decouple	R6			Z		Z2	Z2	Z	
BDI	R++, R--	-1	R0	Z				Z		Z2	Z2
	R+-, R-+	complex	C1		Z		Z		Z		Z
DIII	R++, R--	+1	R4	Z		Z2	Z2	Z			
	R+-, R-+	complex	C1		Z		Z		Z		Z
CII	R++, R--	+1	R6			Z		Z2	Z2	Z	
	R+-, R-+	decouple	R5		Z		Z2	Z2	Z		
CI	R++, R--	-1	R6			Z		Z2	Z2	Z	
	R+-, R-+	decouple	R7				Z		Z2	Z2	Z

具体例

- Topological mirror insulator　　　class AII + mirror

$$\mathcal{H}(k_x, k_y, k_z) = m s_3 + v(k_x\sigma_2 - k_y\sigma_1)s_1 + v_z k_z s_2$$

σ : spin
s : orbital

TR tr. $\hat{T}\Psi(\boldsymbol{k})\hat{T}^{-1} = i\sigma_2\Psi(-\boldsymbol{k}),\ \hat{T}i\hat{T}^{-1} = -i\ \to T = i\sigma_2\mathcal{K}$

yz面でのmirror (= inversion + x軸回りのπ回転)

$$\hat{R}\psi(x,y,z)\hat{R}^{-1} = -i\sigma_1\psi(-x,y,z)\ \to R = -i\sigma_1$$

$$\gamma_1 = \sigma_2 s_1,\ \gamma_2 = -\sigma_1 s_1,\ \gamma_3 = s_2,\ J = i,\ T = i\sigma_2\mathcal{K},\ R = -i\sigma_1$$

class AII + reflection symmetry R : (x,y,z) → (-x,y,z)

$R^2 = -1,\ \ [T, R] = 0 \to R_+$ ⇨ Z (table R1)

● point defect in class D × (mirror + TR)

$$H_{MF} = \frac{1}{2}\sum_{\boldsymbol{k}} \Psi^\dagger_{\boldsymbol{k}} \mathcal{H}(\boldsymbol{k}) \Psi_{\boldsymbol{k}} \qquad \Psi = (\psi_\uparrow, \psi_\downarrow, \psi^\dagger_\uparrow, \psi^\dagger_\downarrow)^T$$

$$\mathcal{H}(k_x, k_y, k_z) = \begin{pmatrix} \varepsilon(\boldsymbol{k}) - \mu + h_x\sigma_1 + h_y\sigma_2 & \frac{\Delta_p}{k_F}\boldsymbol{k}\cdot\boldsymbol{\sigma}(i\sigma_2) \\ (-i\sigma_2)\frac{\Delta_p}{k_F}\boldsymbol{k}\cdot\boldsymbol{\sigma} & -\varepsilon^T(-\boldsymbol{k}) + \mu - h_x\sigma_1 - h_y\sigma_2 \end{pmatrix}$$

$$= \tau_3(\varepsilon(\boldsymbol{k}) - \mu) + \frac{\Delta}{k_F}(-k_x\tau_1\sigma_3 - k_y\tau_2 + k_z\tau_1\sigma_1) + h_x\tau_3\sigma_1 + h_y\tau_3\sigma_1$$

PH tr. $\hat{C}\Psi(\boldsymbol{k})\hat{C}^{-1} = \tau_1\Psi(-\boldsymbol{k}),\ \hat{C}i\hat{C}^{-1} = -i \ \rightarrow C = \tau_1\mathcal{K}$

TR tr. $\hat{T}\Psi(\boldsymbol{k})\hat{T}^{-1} = i\sigma_2\Psi(-\boldsymbol{k}),\ \hat{T}i\hat{T}^{-1} = -i \ \rightarrow T = i\sigma_2\mathcal{K}$

yz面のmirror (= inversion + x軸π回転) $\hat{M}_{yz}\Psi(x,y,z)\hat{M}_{yz}^{-1} = -i\sigma_1\tau_3\Psi(-x,y,z)$

$$\hat{R} := \hat{T}\hat{U}(\pi) \qquad \hat{R}\Psi(x,y,z)\hat{R}^{-1} = i\sigma_1\tau_3\Psi(-x,-y,z),\ \hat{R}i\hat{R}^{-1} = -i \ \rightarrow R = i\sigma_1\tau_3\mathcal{K}$$

$$\gamma_1 = -\tau_1\sigma_3,\ \gamma_2 = -\tau_2,\ \gamma_3 = \tau_1\sigma_1,\ J = i,\ C = \tau_1\mathcal{K},\ R = i\sigma_1\tau_3\mathcal{K}$$

class D + anti-unitary symmetry (x,y,z) → (-x,-y,z)

$R^2 = 1, \quad [C, R] = 0 \rightarrow R_+$ ⇒ Z (table Π2)

● He-B + 面内の磁場　class D + (π-rotation × Time-reversal)

Mizushima-Sato-Machida, PRL (2012).

$$H_{MF}=\frac{1}{2}\sum_{\boldsymbol{k}}\Psi_{\boldsymbol{k}}^{\dagger}\mathcal{H}(\boldsymbol{k})\Psi_{\boldsymbol{k}} \qquad \Psi=(\psi_\uparrow,\psi_\downarrow,\psi_\uparrow^\dagger,\psi_\downarrow^\dagger)^T$$

$$\mathcal{H}(k_x,k_y,k_z)=\begin{pmatrix}\varepsilon(\boldsymbol{k})-\mu+h_x\sigma_1+h_y\sigma_2 & \frac{\Delta_p}{k_F}\boldsymbol{k}\cdot\boldsymbol{\sigma}(i\sigma_2)\\ (-i\sigma_2)\frac{\Delta_p}{k_F}\boldsymbol{k}\cdot\boldsymbol{\sigma} & -\varepsilon^T(-\boldsymbol{k})+\mu-h_x\sigma_1-h_y\sigma_2\end{pmatrix}$$

$$=\tau_3(\varepsilon(\boldsymbol{k})-\mu)+\frac{\Delta}{k_F}(-k_x\tau_1\sigma_3-k_y\tau_2+k_z\tau_1\sigma_1)+h_x\tau_3\sigma_1+h_y\tau_3\sigma_1$$

PH tr. $\hat{C}\Psi(\boldsymbol{k})\hat{C}^{-1}=\tau_1\Psi(-\boldsymbol{k}),\ \hat{C}i\hat{C}^{-1}=-i\ \to C=\tau_1\mathcal{K}$

TR tr. $\hat{T}\Psi(\boldsymbol{k})\hat{T}^{-1}=i\sigma_2\Psi(-\boldsymbol{k}),\ \hat{T}i\hat{T}^{-1}=-i\ \to T=i\sigma_2\mathcal{K}$

xy面内SO(2)$_{L+S}$ のπ-rotation $\hat{U}(\pi)\Psi(x,y,z)\hat{U}(\pi)^{-1}=-i\sigma_3\tau_3\Psi(-x,-y,z)$

$$\hat{R}:=\hat{T}\hat{U}(\pi) \quad \hat{R}\Psi(x,y,z)\hat{R}^{-1}=i\sigma_1\tau_3\Psi(-x,-y,z),\ \hat{R}i\hat{R}^{-1}=-i\ \to R=i\sigma_1\tau_3\mathcal{K}$$

$$\gamma_1=-\tau_1\sigma_3,\ \gamma_2=-\tau_2,\ \gamma_3=\tau_1\sigma_1,\ J=i,\ C=\tau_1\mathcal{K},\ R=i\sigma_1\tau_3\mathcal{K}$$

class D + anti-unitary symmetry (x,y,z) → (-x,-y,z)

$R^2=1,\quad [C,R]=0\to R_+$ ⟹ Z (table Π2)

Classification of defect zero mode with a unitary/anti-unitary π-rotation symmetry

$$R\ :\ (x_1, x_2, x_3, \dots) \to (-x_1, -x_2, x_3, \dots)$$

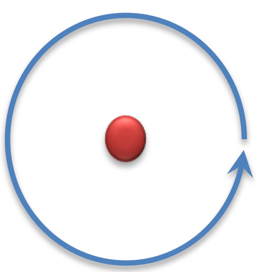
座標を2つ反転

任意次元のdefectを考える

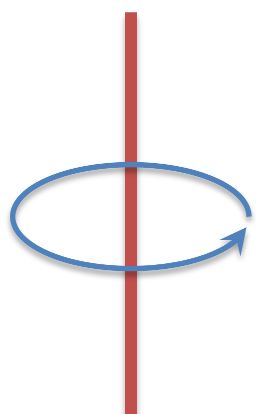
- point defect (δ-1 = 0)
- line defect (δ-1 = 1)
- membrane defect (δ-1 = 2)
- ・・・

d : dimension
δ-1 : defect dimension
D : co-dimension of defect
δ = d - D

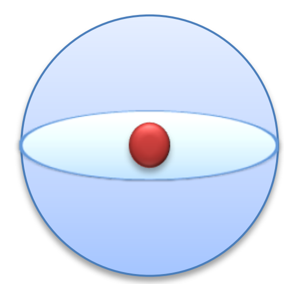
$$(s_1, s_2, \ldots, s_D)$$: parameters surrounding a defect



$d = 2, \delta - 1 = 0, (s_1)$



$d = 3, \delta - 1 = 1, (s_1)$



$d = 3, \delta - 1 = 0, (s_1, s_2)$

D : co-dimension of defect

$$H_{d,D} = \sum_{i=1}^{d} \gamma_i k_i + \sum_{a=1}^{D} \tilde{\gamma}_a s_a + H$$

$$\gamma_i^2 = \tilde{\gamma}_a^2 = 1$$

$$\{H, \Gamma\} = [H, J] = [H, T] = \{H, C\} = \{J, T\} = \{J, C\} = [T, C] = 0$$

$$\{\gamma_i, H\} = [\gamma_i, J] = \{\gamma_i, T\} = [\gamma_i, C] = \{\gamma_i, \Gamma\} = 0$$

$$\{\tilde{\gamma}_a, H\} = [\tilde{\gamma}_a, J] = [\tilde{\gamma}_a, T] = \{\tilde{\gamma}_a, C\} = \{\tilde{\gamma}_a, \Gamma\} = 0$$

D次元球面のパラメタ付け

$$\hat{\boldsymbol{n}} = (n_1, n_2, \ldots, n_{D+1})$$

$$\begin{aligned}
n_1 &= \sin\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4 \cdots \sin\theta_D \\
n_2 &= \cos\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4 \cdots \sin\theta_D \\
n_3 &= \cos\theta_2 \sin\theta_3 \sin\theta_4 \cdots \sin\theta_D \\
n_4 &= \cos\theta_3 \sin\theta_4 \cdots \sin\theta_D \\
n_5 &= \cos\theta_4 \cdots \sin\theta_D \\
&\ldots \\
n_D &= \cos\theta_{D-1} \sin\theta_D \\
n_{D+1} &= \cos\theta_D
\end{aligned}$$

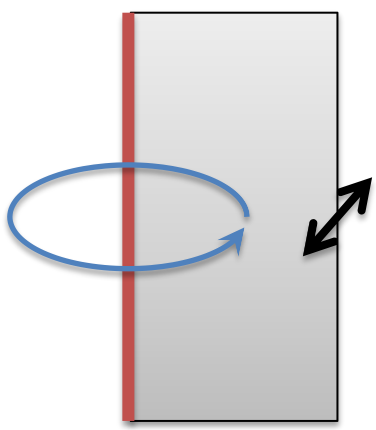
$$(n_1, n_2, n_3, \ldots) \to (-n_1, n_2, n_3, \ldots) \ : \ \theta_1 \to -\theta_1$$
$$(n_1, n_2, n_3, \ldots) \to (n_1, -n_2, n_3, \ldots) \ : \ \theta_1 \to \pi - \theta_1$$
$$(n_1, n_2, n_3, \ldots) \to (n_1, n_2, -n_3, \ldots) \ : \ \theta_1 \to \pi - \theta_2$$

- defectを囲むパラメタへのπ-rotationの作用に仕方は3通りある

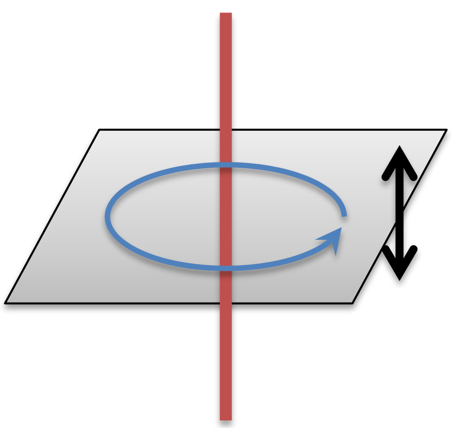
(I) π-rotationを受ける座標が$(n_1, n_2, \ldots, n_{D+1})$に入っている
→ひとつだけ反転：$(s_1,s_2,\ldots,s_D) \to (-s_1,s_2,\ldots,s_D)$

(II) π-rotationを受ける座標が$(n_1, n_2, \ldots, n_{D+1})$に入っていない
→反転しない：$(s_1,s_2,\ldots,s_D) \to (s_1,s_2,\ldots,s_D)$

- point defectのみ特殊で、タイプ(I)のみ



type I



type II

Unitary π-rotation, type (I)

D : co-dimension of defect

$$H_{d,D} = \gamma_1 k_1 + \sum_{i=2}^{d} \gamma_i k_i + \tilde{\gamma}_1 s_1 + \sum_{a=2}^{D} \tilde{\gamma}_a s_a + H$$

$$\gamma_i^2 = \tilde{\gamma}_a^2 = 1$$

$$\{H, \Gamma\} = [H, J] = [H, T] = \{H, C\} = \{J, T\} = \{J, C\} = [T, C] = 0$$

$$\{\gamma_i, H\} = [\gamma_i, J] = \{\gamma_i, T\} = [\gamma_i, C] = \{\gamma_i, \Gamma\} = 0$$

$$\{\tilde{\gamma}_a, H\} = [\tilde{\gamma}_a, J] = [\tilde{\gamma}_a, T] = \{\tilde{\gamma}_a, C\} = \{\tilde{\gamma}_a, \Gamma\} = 0$$

$$\{R, \gamma_1\} = [R, \gamma_2] = \cdots = [R, \gamma_d] = 0$$

$$\{R, \tilde{\gamma}_1\} = [R, \tilde{\gamma}_2] = \cdots = [R, \tilde{\gamma}_D] = 0$$

$$R^2 = 1$$

付録

$$Cl_{p,q} \to Cl_{p,q+1} \quad : \quad R_{q-p}$$

$$Cl_{p,q} \to Cl_{p+1,q} \quad : \quad R_{p+2-q}$$

(a) complex classes

q	Cl_q	C_q	$\pi_0(C_q)$
0	$\mathbb{C}$	$(U(n+m)/U(n)\times U(m))\times\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{C}\oplus\mathbb{C}$	$U(n)$	0

(b) real classes

q	$Cl_{0,q}$	R_q	$\pi_0(R_q)$
0	$\mathbb{R}$	$(O(n+m)/O(n)\times O(m))\times\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{R}\oplus\mathbb{R}$	$O(n)$	$\mathbb{Z}_2$
2	$\mathbb{R}(2)$	$O(2n)/U(n)$	$\mathbb{Z}_2$
3	$\mathbb{C}(2)$	$U(2n)/Sp(n)$	0
4	$\mathbb{H}(2)$	$(Sp(n+m)/Sp(n)\times Sp(m))\times\mathbb{Z}$	$\mathbb{Z}$
5	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$Sp(n)$	0
6	$\mathbb{H}(4)$	$Sp(n)/U(n)$	0
7	$\mathbb{C}(8)$	$U(n)/O(n)$	0

class	TRS	PHS	chiral	C_q or R_q	d=0	d=1	d=2	d=3	d=4	d=5	d=6	d=7
A				C0	Z		Z		Z		Z	
AIII			1	C1		Z		Z		Z		Z
AI	1			R0	Z				Z		Z2	Z2
BDI	1	1		R1	Z2	Z				Z		Z2
D		1		R2	Z2	Z2	Z				Z	
DIII	-1	1		R3		Z2	Z2	Z				Z
AII	-1			R4	Z		Z2	Z2	Z			
CII	-1	-1		R5		Z		Z2	Z2	Z		
C		-1		R6			Z		Z2	Z2	Z	
CI	1	-1		R7				Z		Z2	Z2	Z

Reference

Kitaev, A. (2009). Periodic table for topological insulators and superconductors. In *AIP Conference Proceedings* (Vol. 22, pp. 22–30). AIP. doi:10.1063/1.3149495

Teo, J., & Kane, C. (2010). Topological defects and gapless modes in insulators and superconductors. *Physical Review B*. Retrieved from http://prb.aps.org/abstract/PRB/v82/i11/e115120

Chiu, C., Yao, H., & Ryu, S. (2013). Classification of topological insulators and superconductors in the presence of reflection symmetry. Retrieved from http://arxiv.org/abs/1303.1843

Morimoto, T., & Furusaki, A. (2013). Topological classification with additional symmetries from Clifford algebras. Retrieved from http://arxiv.org/abs/1306.2505v2