Non-Hermitian Matrix Topology: Symmetry, Gap Conditions, and Classification

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- A supplemental Mathematica notebook is available at this link.

Outline

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   — Equivalence relation in general
   — Topology for matrices

3. Symmetry classes
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4. Topological classification
   — Hermitianization and flattening
   — Classifying space
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5. Intrinsic non-Hermitian topology

Introduction: Overview of one-particle non-Hermitian systems

Non-Hermitian Systems

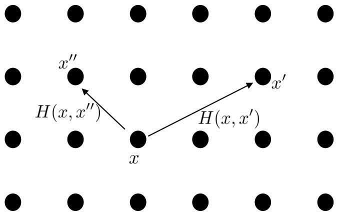
- Non-Hermitian Hamiltonians and matrices often appear in many physical systems.
- These include Photonics, Mechanics, Electrical Circuits, Biological Physics, Optomechanics, Hydrodynamics, Open Quantum Systems, and Non-unitary Conformal Field Theories.
- For more details on where non-Hermiticity shows up, see the review by, for example, [Ashida=Gong=Ueda, 2006.01837].

One-particle non-Hermitian Systems

- In this lecture, I will provide a brief introduction to the topological aspects of *one-particle* non-Hermitian systems. Specifically, we'll delve into the topological nature of matrices

$$H = \{H_{\sigma\sigma'}(x, x')\}_{x,x'\in\Lambda,\sigma,\sigma'=1,\dots,N}$$

  defined over a d-dimensional lattice, Λ, with internal degrees of freedom given by $\sigma = 1, \dots, N$.
- We'll assume the hopping range is local, i.e., $||H(x, x')|| < e^{-|x-x'|/\xi}$. (Otherwise, the concept of "dimension" would be meaningless.)
- Each physical system might possess intrinsic internal symmetries (which do not affect spatial positions).
- We may be interested in the physics robust against the disorder effect, which is compatible only with the internal symmetry.



Example: Wilson Dirac Operator

- In lattice gauge theory, we examine the lattice Dirac operator on the Euclidean cubic lattice. The Wilson Dirac operator is defined as:

$$D_W[U] = I - \kappa \sum_{\nu=1}^{3} \left[(I+\gamma_\nu) T_{\nu+} + (I-\gamma_\nu) T_{\nu-} \right] - \kappa \left[e^{\mu} (I+\gamma_4) T_{4+} + e^{-\mu} (I-\gamma_4) T_{4-} \right],$$

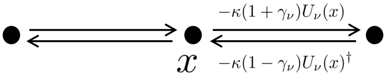
where:

$$[T_{\nu+}]_{x,y} = U_\nu(x) \delta_{x+\hat{\nu},y}, \quad [T_{\nu-}]_{x,y} = U_\nu(y)^\dagger \delta_{x-\hat{\nu},y}.$$

Here, $U_\mu(x) \in U(N)$ represents the $U(N)$ gauge field, and μ denotes the chemical potential.

- When the chemical potential μ is absent (i.e., $\mu = 0$), D_W satisfies the γ_5-Hermiticity condition:

$$\gamma_5 D_W[U]^\dagger \gamma_5 = D_W[U].$$



Ex. Mechanical Metamaterials

- Consider a mass-spring model with the equation of motion:

$$\ddot{\boldsymbol{u}} = -D\boldsymbol{u} + \Gamma\dot{\boldsymbol{u}},$$

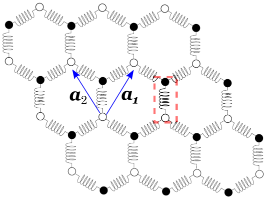
where $\boldsymbol{u} = \{u_i(x)\}_{x,i}$ denotes the displacement vector components.

- The matrices D and Γ are real with D being positive semi-definite for system stability.
- Without friction, Γ is skew-symmetric (i.e., $\Gamma^T = -\Gamma$). However, this isn't generally the case.
- Using the variable $\tilde{\boldsymbol{u}} = (\sqrt{D}\boldsymbol{u}, i\dot{\boldsymbol{u}})^T$, the dynamics follows a Schrödinger-type equation [Kane=Lubensky 1308.0554, Süsstrunk=Huber 1604.01033.]:

$$i\frac{d}{dt}\tilde{\boldsymbol{u}} = H\tilde{\boldsymbol{u}}, \quad H = \begin{pmatrix} O & \sqrt{D} \\ \sqrt{D} & i\Gamma \end{pmatrix}.$$

- The Hamiltonian H inherently exhibits particle-hole symmetry:

$$\sigma_z H^* \sigma_z = -H.$$



[Figure from Yoshida=Hatsugai, PRB **100**, 054109 (2019)]

Some characteristics of Non-Hermitian Matrices

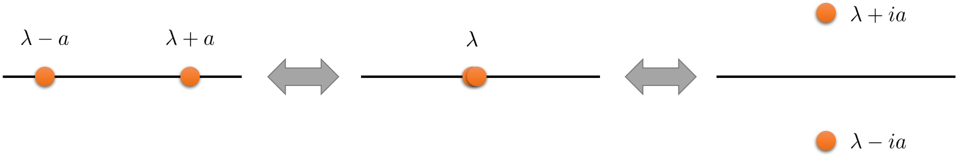
- Eigenvalues can be complex.
- Exceptional Points: These occur when the dimension of the Jordan block is 2 or more, making the matrix H non-diagonalizable. Example matrices include:

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

- Non-Hermitian Skin Effect [Yao=Wang 1803.01876]: The matrix behavior is sensitive to different boundary conditions, such as periodic boundary condition (PBC), open boundary condition (OBC), and semi-infinite boundary condition, among others.

PT Symmetry Breaking Bender=Boettcher physics/9712001

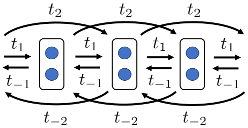
- For matrices with PT-symmetry, represented by $H^* = H$, eigenvalues either appear as an isolated real value, $E^* = E$, or as a conjugate pair, (E, E^*).
- PT-symmetry breaking refers to the transition where two real eigenvalues merge to form a complex conjugate pair (E, E^*), or vice versa. Such transitions occur at an exceptional point.

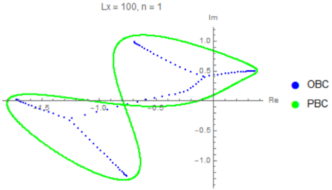


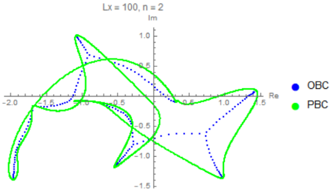
$$H = \begin{bmatrix} \lambda & a \\ a & \lambda \end{bmatrix} \qquad H = \begin{bmatrix} \lambda & a \\ 0 & \lambda \end{bmatrix} \qquad H = \begin{bmatrix} \lambda & a \\ -a & \lambda \end{bmatrix}$$

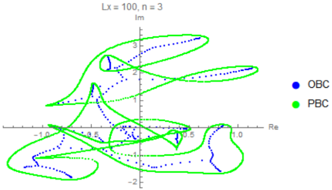
PBC vs OBC

Here are some spectra of 1-dimensional non-Hermitian models.







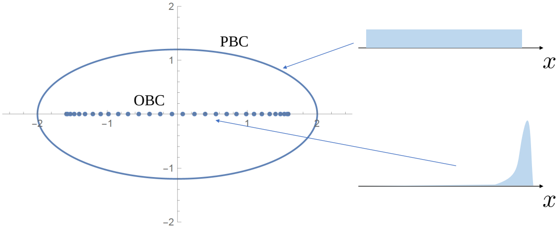


Non-Hermitian Skin effect Yao=Wang 1803.01876

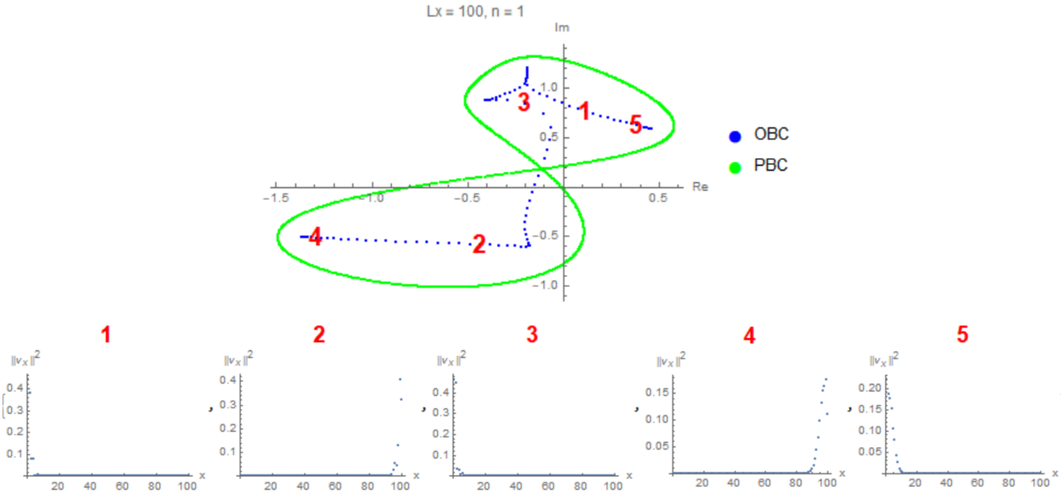
- PBC $\neq$ OBC for spectra. Extreme sensitivity against the boundary condition.
- In OBC, $O(L)$ modes are localized at an edge.
- A prime example is the Hatano-Nelson model, a one-dimensional model with non-reciprocal hopping.
- Non-Hermitian Skin effect has a topological origin. [Zhang=Yang=Fang 1910.01131, Okuma=Kawabata=KS=Sato 1910.02878] ($\to$ Okuma-san's lecture)

$$H = \sum_{x\in\mathbb{Z}} te^{g} f_{x+1}^\dagger f_x + te^{-g} f_x^\dagger f_{x+1} \quad \overset{\text{PBC}}{\Longrightarrow} \quad H_{\text{PBC}} = \sum_k f_k^\dagger (te^{g} e^{-ik} + te^{-g} e^{ik}) f_k,$$

$$\overset{\text{OBC}}{\Longrightarrow} \quad H_{\text{OBC}} = \sum_{x=1}^{L} t\tilde{f}_{x+1}^\dagger \tilde{f}_x + t\tilde{f}_x^\dagger \tilde{f}_{x+1}, \quad \tilde{f}_x^\dagger = e^{gx} f_x^\dagger$$

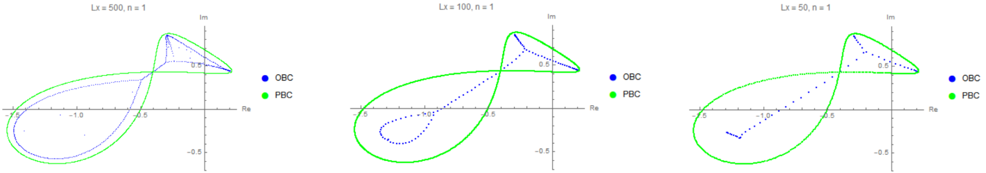


Example: No symmetry



Numerical Rounding Error is not Negligible

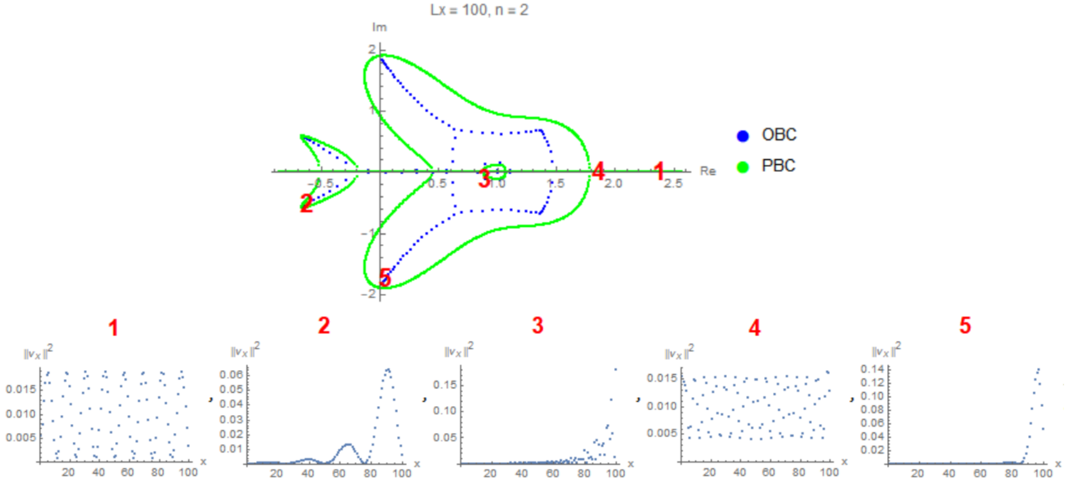
- In computational calculation, *rounding error* refers to the small differences between the actual real number and its nearest representable value in the computer.
- Since $O(L)$ skin modes are exponentially localized at an edge, these small differences can significantly affect the results.



- The "Non-Bloch band theory" is used to compute the OBC spectrum in the thermodynamic limit. Yao=Wang 1803.01876, Yokomizo=Murakami 1902.10958

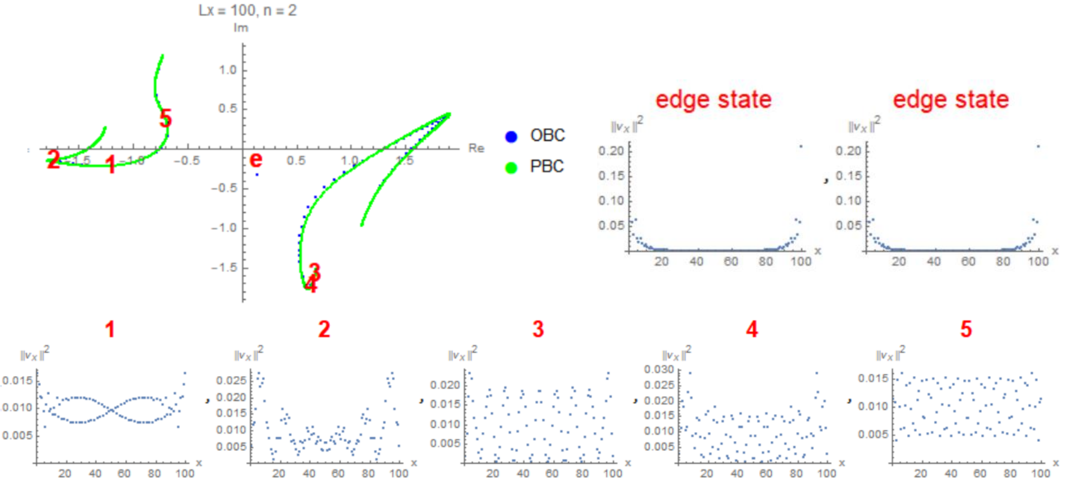
Example: Pseudo Hermiticity

$$\eta t_n^\dagger \eta^\dagger = t_{-n}, \quad \eta^2 = 1, \quad \mathrm{tr}\left[\eta\right] = 0.$$



Example: Inversion symmetry → the Non-Hermitian skin effect is suppressed

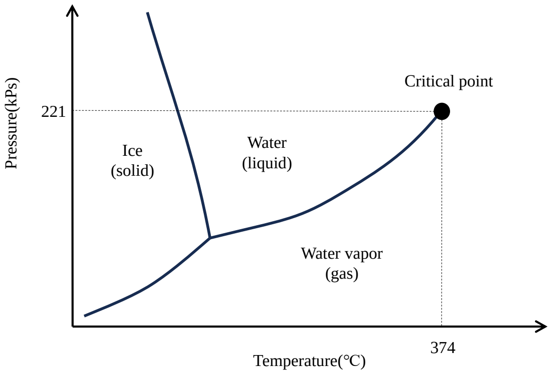
$$u t_n u^\dagger = t_{-n}, \quad u^2 = 1.$$



Gap Conditions and Topology

Equivalence Condition and Phases of Matter

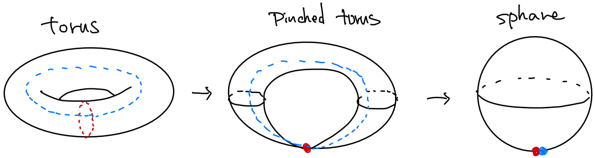
- Water Phase Diagram:



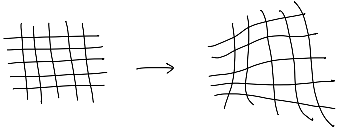
- The ice and water phases are distinct: A singularity in the thermodynamic function exists between these two phases, indicating a phase transition.
- Conversely, water and vapor can be considered the same phase since there exists a continuous path connecting them without encountering a thermodynamic singularity.

Topological Equivalence

- A torus and a sphere are considered to have distinct topologies.
- By shrinking one circle of the torus, we obtain a pinched torus. By further shrinking another circle, we ultimately transform it into a sphere.



- What exactly defines topology?
- Topological equivalence is determined by deformations that preserve the local structure of the Euclidean space.



- Given a defined equivalence relation, we can identify a set of equivalence classes.

Topology of Matrices

- What does it mean to classify matrices topologically?
- Consider two $N \times N$ matrices H_0 and H_1.
- They can be connected to each other by a continuous path defined as:

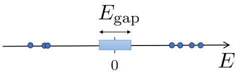
$$H_t = (1-t)H_0 + tH_1, \quad t \in [0,1].$$

$\rightarrow$ no topological classification.

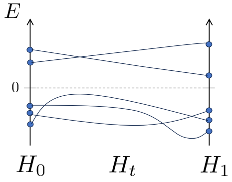
Hermitian Matrices: Gap Condition

- For meaningful classifications, we impose a gap condition.
- For Hermitian matrices H (where $H^\dagger = H$), the eigenvalues E are always real $E \in \mathbb{R}$.
- A reasonable gap condition is a finite energy gap $E_{\text{gap}} > 0$ around zero (or the Fermi energy E_F):

$$E \neq 0.$$



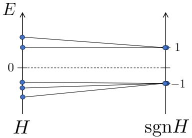
- Two Hermitian matrices H_0 and H_1 with no zero eigenvalues are considered equivalent if they can be continuously connected via a homotopy $H_{t\in[0,1]}$ provided that H_t also satisfies the gap condition throughout.



Hermitian Matrices: Gap Condition (cont.)

- We may think two H_0 and H_1 are equivalent if the numbers of negative eigenstates are the same.
- This is true. H can be flattened while keeping the gap condition.

$$H_t = \{(1-t)E_n + t\,\mathrm{sgn}(E_n)\}\left|n\right\rangle\left\langle n\right| \xrightarrow{t\to 1} \sum_{n=1}^{N} \mathrm{sgn}(E_n)\left|n\right\rangle\left\langle n\right| =: \mathrm{sgn}H.$$



- The flattened Hamiltonian $\mathrm{sgn}H$ is uniquely identified with a point of the complex Grassmaniann:

$$\mathrm{sgn}H = U\begin{pmatrix} 1_{N-M} & \\ & -1_M \end{pmatrix}U^\dagger, \quad U \sim U\begin{pmatrix} V & \\ & W \end{pmatrix},$$

$$U \in U(N), V \in U(N-M), W \in U(M).$$

$$\to H \in \mathrm{Gr}_M(\mathbb{C}^N) = U(N)/U(N-M)\times U(M).$$

- No further classifications arise since the complex Grassmaniann is simply connected $\pi_0[\mathrm{Gr}_M(\mathbb{C}^N)] = 0$. For example, $\mathrm{Gr}_1(\mathbb{C}^2) \cong S^2$.

Hermitian Matrices: Example of Symmetry

- Even when two matrices have an equal number of negative (and positive) eigenvalues, certain symmetries can forbid a continuous transformation between them.
- Let's consider a Hermitian matrix H with an additional skew-symmetric constraint

$$H^T = -H, \quad H \in \mathrm{Mat}_{2N\times 2N}(\mathbb{C}).$$

- The Pfaffian $\mathrm{pf}\, H \in \mathbb{C}$ is a well-defined. [1]
- Given the relationship $(\mathrm{pf}\, H)^* = \mathrm{pf}\, H^* = \mathrm{pf}\, H^T = (-1)^N \mathrm{pf}\, H$, the ratio of the Pfaffians of two matrices is always real:

$$\frac{\mathrm{pf}\, H_0}{\mathrm{pf}\, H_1} \in \mathbb{R},$$

implying that its sign is an invariant that takes on values in $\mathbb{Z}_2 = \{\pm 1\}$.
- For example, consider these two matrices:

$$H_0 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad H_1 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

No continuous transformation connects them while preserving the gap condition and the symmetries $H^\dagger = H$ and $H^T = -H$.

[1] $\mathrm{pf}\, H := \sum_{\sigma\in S_{2N}, \sigma(2i-1)<\sigma(2i), \sigma(1)<\sigma(3)<\cdots<\sigma(2N-1)} \mathrm{sgn}(\sigma) A_{\sigma(1)\sigma(2)} \cdots A_{\sigma(2N-1)\sigma(2N)}$

Hermitian Matrices: Finite Space dimensions & Translational Invariance

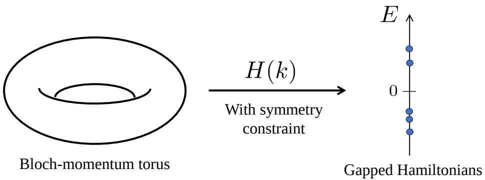
- We have discussed Hermitian matrices H without an extended space direction.
- In a d-dimensional finite space, the legs of H extend to an infinite lattice:

$$H = H(x, x'), \quad x, x' \in \mathbb{Z}^d.$$

- Translational symmetry lets us define the Hamiltonian in the Bloch-momentum torus T^d:

$$H(x, x') = H(x - x') = \sum_{k \in T^d} H(k) e^{ik\cdot(x-x')}.$$

- Classification is about homotopy for matrix families $H(k)$ over torus T^d.



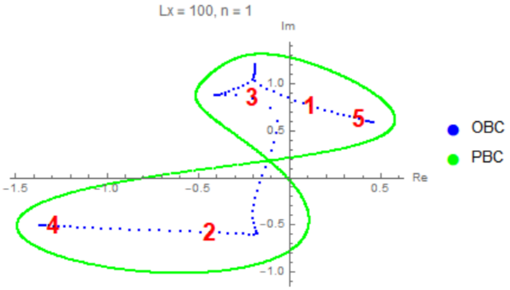
- $H_0(k)$ is equivalent to $H_1(k)$ if a homotopy $H_{t\in[0,1]}(k)$ exists that bridges them while preserving the gap condition and symmetry.

Non-Hermitian Matrices: What is the Gap Condition

- Eigenvalues of non-Hermitian matrices are complex.
- What is a meaningful gap condition?
- A characteristic feature of complex eigenvalues is that in a PBC, the phase of an eigenvalue around a reference energy E_{ref} may have a winding number

$$W(E_{\mathrm{ref}}) = \frac{1}{2\pi i} \oint d \log \det[H_{\mathrm{PBC}}(k) - E_{\mathrm{ref}}] \in \mathbb{Z}.$$

$\to$ the origin of the non-Hermitian skin effect [Zhang-Yang-Fang 1910.01131, Okuma-Kawabata-KS-Sato 1910.02878].



Non-Hermitian Matrices: Point Gap

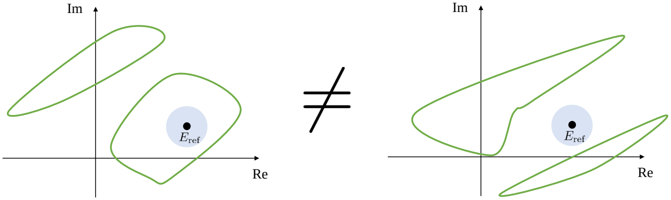
Gong-Ashida-Kawabata-Takasan-Higashikawa-Ueda 1802.07964

- The winding number $W(E_{\mathrm{ref}})$ is stable unless an eigenvalue touches the reference energy E_{ref}.
- The point gap condition

$$E \neq E_{\mathrm{ref}} \quad (\det(H(k) - E_{\mathrm{ref}}) \neq 0)$$

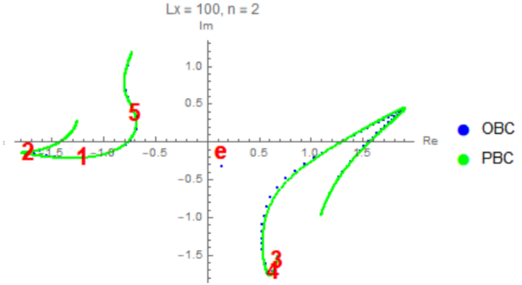
  makes sense.

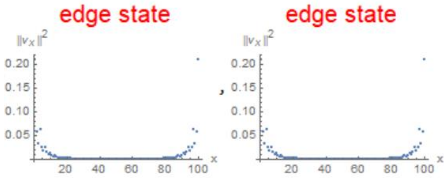
- Eg: The following two Hamiltonians are in distinct point-gapped topological phases w.r.t. the reference energy E_{ref}.



Non-Hermitian Matrices: Remnants of Hermitian edge states

- Even with non-Hermiticity, the remnant of Hermitian topological phases, the boundary states, might persist.
- A minor perturbation doesn't eliminate the edge states inherent to Hermitian topological phases. This is because the spectrum can deform continuously smoothly when perturbed slightly.



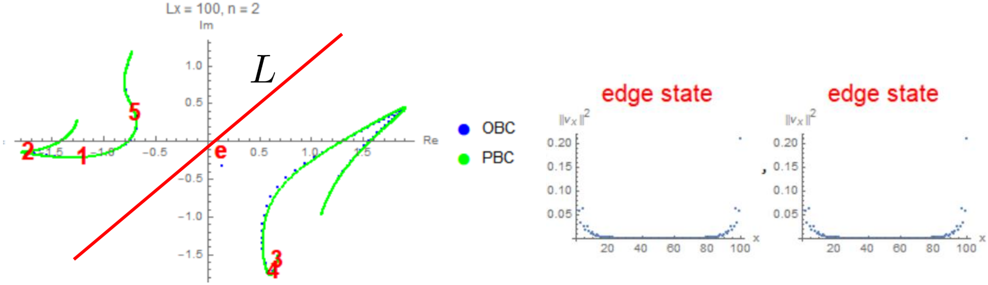


Non-Hermitian Matrices: Line Gap Kawabata-KS-Ueda-Sato 1812.09133

- To capture such remnants of Hermitian topological edge states in a non-Hermitian system, we introduce the concept of a line gap:

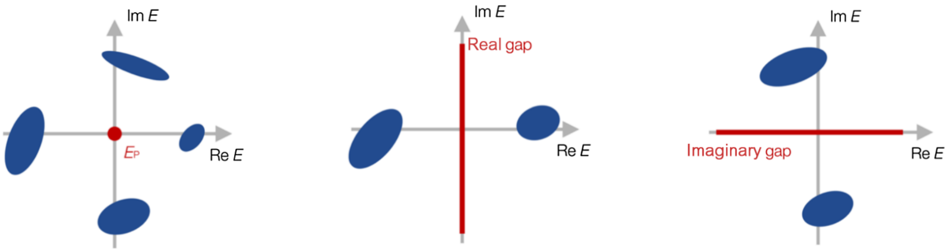
$$\mathrm{Spec}(H) \cap L = \emptyset, \quad \text{where } L \text{ is a line in the complex plane } \mathbb{C}.$$

- Hamiltonians $H_0(k)$ and $H_1(k)$ are considered to belong to the same topological phase with respect to the line gap if there exists a homotopy $H_{t\in[0,1]}(k)$ that connects them while preserving the line gap and the associated symmetry.



Non-Hermitian Matrices: Point Gap and Line gap

- It is useful to introduce two types of line gaps: real line gap and imaginary line gap. These are consistent with symmetries associating E with $-E, E^*$, or $-E^*$ (detailed later).
- P: Point-gap $\quad E - E_{\mathrm{ref}} \neq 0.$
- $\mathrm{L_r}$: Real line gap $\quad \mathrm{Re}(E - E_{\mathrm{ref}}) \neq 0.$
- $\mathrm{L_i}$: Imaginary line gap $\quad \mathrm{Im}(E - E_{\mathrm{ref}}) \neq 0.$



[Figure from Kawabata=KS=Ueda=Sato 1812.09133]

Symmetry in non-Hermitian systems

Symmetries in Non-Hermitian Systems

- What kind of symmetries exist in non-Hermitian systems?
- Example:
  - Time-reversal symmetry (TRS) is a fundamental symmetry.

$$U_T H^* U_T^\dagger = H.$$

  - In the mean-field approach to superconductors, the Bogoliubov–de Gennes (BdG) Hamiltonian H_{BdG} inherently possesses particle-hole symmetry (PHS).[2]

$$U_C H_{\mathrm{BdG}}^T U_C^\dagger = -H_{\mathrm{BdG}}, \quad H_{\mathrm{BdG}} = \begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^T \end{pmatrix}, \quad U_C = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

  - Bosonic systems with quadratic interactions are captured by the bosonic BdG Hamiltonian $\hat{H} = \frac{1}{2}(\boldsymbol{a}^\dagger, \boldsymbol{a}) H_{\mathrm{BdG}} (\boldsymbol{a}, \boldsymbol{a}^\dagger)^T$. To maintain the bosonic commutation relation, H_{BdG} must be diagonalized using a paraunitary matrix [3], which is the same as the standard diagonalization of the effective matrix $H_{\sigma\mathrm{BdG}} = \sigma_z H_{\mathrm{BdG}}$. While $H_{\sigma\mathrm{BdG}}$ is non-Hermitian, the Hermiticity of $\hat{H}$ is encoded in its pseudo-Hermiticity:

$$\sigma_z H_{\sigma\mathrm{BdG}}^\dagger \sigma_z = H_{\sigma\mathrm{BdG}}.$$

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[2] Note that $\Delta^T = -\Delta$ due to the fermion anti-commutation relation.
[3] $U\sigma_z U^\dagger = \sigma_z, U^\dagger \sigma_z U = \sigma_z$.

Symmetries in Non-Hermitian Systems (cont.)

- We consider the following 8 types of symmetries :

Symmetry in non-Hermitian systems

$$u \left\{ \begin{array}{c} H \\ H^* \\ H^T \\ H^\dagger \end{array} \right\} u^\dagger = \left\{ \begin{array}{c} H \\ -H \end{array} \right\}, \quad u \text{ is a unitary matrix.}$$

- This choice is ad hoc. In quantum mechanics, Winger's theorem tells us symmetry, a transformation that does not change the observation, is either unitary or anti-unitary. In non-Hermitan systems without specifying a physical system, we have no such guiding principles. We may consider different types of symmetry such as

$$u \left\{ \begin{array}{c} H \\ H^* \\ H^T \\ H^\dagger \end{array} \right\} v^\dagger = e^{i\phi} H, \quad u \neq v, \quad e^{i\phi} \in U(1).$$

For example, the symmetry type $uH^\dagger v^\dagger = H$ was discussed to construct the symmetry indicator in KS=Ono 2105.00677.

Symmetries in Non-Hermitian Systems (cont.)

- Let G be a group. We introduce three homomorphims [4] $\phi, \eta, c : G \to \mathbb{Z}_2 = \{\pm 1\}$ to specify the type of symmetry as

$$\left\{ \begin{array}{ll} u_g H u_g^\dagger & (\phi_g = 1, \eta_g = 1) \\ u_g H^* u_g^\dagger & (\phi_g = -1, \eta_g = 1) \\ u_g H^T u_g^\dagger & (\phi_g = -1, \eta_g = -1) \\ u_g H^\dagger u_g^\dagger & (\phi_g = 1, \eta_g = -1) \end{array} \right\} = c_g H, \quad g \in G,$$

- Comparing the transformation with two consecutive h, g transformations and the transformation with gh, we have

$$\left\{ \begin{array}{ll} u_g u_h & (\phi_g = 1) \\ u_g u_h^* & (\phi_g = -1) \end{array} \right\} = z_{g,h} u_{gh}, \quad z_{g,h} \in U(1), \quad g, h \in G.$$

- The relation $(gh)k = g(hk)$ gives the constraint relations

$$z_{h,k}^{\phi_g} z_{gh,k}^{-1} z_{g,hk} z_{g,h}^{-1} = 1, \quad g, h, k \in G.$$

(This means $z = (z_{g,h})$ is a two-cycle in $Z^2(G, U(1)_\phi)$.)

[4] Let G_0 and G_1 be groups. $f : G_0 \to G_1$ is said to be a homomorphism if $f(gh) = f(g)f(h)$ is met.

8 types of symmetries (names from Kawabata-KS-Ueda-Sato 1812.09133)

ϕ_g	η_g	c_g	Sym.	Energy constraints	Name
1	1	1	$u_g H u_g^\dagger = H$	$E \to E$	Unitary
1	-1	1	$u_g H^\dagger u_g^\dagger = H$	$E \to E^*$	Pseudo Hermiticity (PH)
-1	1	1	$u_g H^* u_g^\dagger = H$	$E \to E^*$	Time-reversal symmetry (TRS)
-1	-1	1	$u_g H^T u_g^\dagger = H$	$E \to E$	Time-reversal dagger symmetry (TRS†)
-1	1	-1	$u_g H^* u_g^\dagger = -H$	$E \to -E^*$	Particle-hole dagger symmetry (PHS†)
-1	-1	-1	$u_g H^T u_g^\dagger = -H$	$E \to -E$	Particle-hole symmetry (PHS)
1	1	-1	$u_g H u_g^\dagger = -H$	$E \to -E$	Sublattice symmetry (SLS)
1	-1	-1	$u_g H^\dagger u_g^\dagger = -H$	$E \to -E^*$	Chiral symmetry (CS)

and finer classifications (detailed from the next slide).

38 symmetry classes Kawabata-KS-Ueda-Sato 1812.09133

- What are fundamentally different symmetry classes that govern the topological nature of matrices?
  $\rightarrow$ We eventually reach the 38 symmetry classes. (cf. 10 Altland-Zirnbauer symmetry classes in Hermitian systems. cond-mat/9602137)

Proof

(i) The Hamiltonian H is block-diagonalized to the irreducible representations $\alpha, \beta, \gamma, \dots$ of the unitary subgroup $G_0 = \{g \in G | \phi_g = \eta_g = c_g = 1\} \subset G$.

$$H = \begin{pmatrix} H_\alpha & & & \\ & H_\beta & & \\ & & H_\gamma & \\ & & & \ddots \end{pmatrix}$$

(ii) A group element $g \in G$ in which either ϕ_g, η_g, or c_g is -1, acts on each block H_α as either

- g preserves the irreducible representation α. g is closed inside the block H_α.
  $\rightarrow$ g acts as a $\mathbb{Z}_2$ symmetry inside the block H_α. (cf. Wigner criteria)
- g exchanges the irreducible representations $H_\alpha \overset{g}{\longleftrightarrow} H_\beta$.
  $\rightarrow$ H_β is just a copy of H_α. The topological nature is determined only in the block H_α.

38 symmetry classes (cont.)

(iii) The problem is recast as how different symmetry actions there are in a single block H_α.

(iv) We can assume the absence of unitary symmetry (i.e., $(\phi_g, \eta_g, c_g) \neq (1,1,1)$).
$\rightarrow$ The symmetry group G realized in the single block is either one of

$$G = \mathbb{Z}_2^{\times N}, \quad N = 0, 1, 2, 3.$$

(Otherwise, there is a unitary group element.)

(v) For a group element g with $\phi_g = -1$, namely antiunitary symmetry, the square is proportional to identity (since $g^2 = e$) but its coefficient is quantized to a sign [5]

$$u_g u_g^* = \pm 1.$$

[5] The coefficient should be a sign: Set $u_g u_g^* = e^{\imath\phi}$. Then, $e^{i\phi} u_g = u_g u_g^* u_g = u_g (u_g u_g^*)^* = u_g e^{-i\phi}$. The sign ± 1 is unchanged under $u_g \mapsto e^{i\alpha} u_g$.

38 symmetry classes (cont.)

(vi) Case of $N = 0$ — Unique.

(vii) Case of $N = 1$ — Seven patterns:

$$(\phi_1, \eta_1, c_1) = (-1, 1, 1), (-1, -1, 1), (-1, 1, -1), (-1, -1, -1), (1, -1, 1), (1, 1, -1), (1, -1, -1).$$

For $\phi_1 = -1$, we have 2 cases for each, resulting in $2 \times 4 + 3 = 11$.

(viii) Case of $N = 2$ — When $\phi_g = -1$ is included, there are four patterns

$$\begin{aligned}\{(\phi_1, \eta_1, c_1), (\phi_2, \eta_2, c_2)\} =& \{(-1, 1, 1), (-1, -1, 1)\}, \{(-1, 1, 1), (-1, 1, -1)\}, \\ & \{(-1, 1, 1), (-1, -1, -1)\}, \{(-1, -1, 1), (-1, 1, -1)\},\end{aligned}$$

and choices of the signs of $u_1 u_1^* = \pm 1$ and $u_2 u_2^* = \pm 1$ for each. When $\phi_g = -1$ is not included, there is only one pattern

$$\{(\phi_1, \eta_1, c_1), (\phi_2, \eta_2, c_2)\} = \{(1, -1, 1), (1, 1, -1)\},$$

with the commutation or anticommutation relation of them $u_1 u_2 = \pm u_2 u_1$. As a result, we have $4 \times 4 + 2 = 18$.

38 symmetry classes (cont.)

(ix) Case of $N = 3$ — The set of three generators is unique

$$\{(\phi_1, \eta_1, c_1), (\phi_2, \eta_2, c_2), (\phi_3, \eta_3, c_3)\} = \{(-1, 1, 1), (-1, -1, 1), (-1, 1, -1)\}.$$

The choices of the signs of $u_1 u_1^* = \pm 1$, $u_2 u_2^* = \pm 1$, and $u_3 u_3^* = \pm 1$. We have $2 \times 2 \times 2 = 8$.

(x) In sum,

$$1 + 11 + 18 + 8 = 38 \text{ classes.} \quad \square$$

- Cf. This is contrasted to the 43-fold classes in the pioneered work by Bernard-LeClair. [cond-mat/0110649] This is due to overcounting and overlooking.

Issues in the 38 Symmetry Classes of Non-Hermitian Systems

- Having fundamental symmetry classes, several fundamental issues arise:
  - **Anderson localization problem** Hatano=Nelson cond-mat/9603165, ...
  - **Spectral statistics** (Level-spacing distribution) of random matrices Hamazaki=Kawabata=Kura=Ueda 1904.13082, ...
  - **Topological classification** w.r.t. gap conditions (point or line gap) Gong=Ashida=Kawabata=Takasan=Higashikawa=Ueda 1802.07964, Kawabata=KS=Ueda=Sato 1812.09133, Zhou=Lee 1812.10490, ...
  - **Symmetry protected exceptional points?** Kawabata=Bessho=Sato 1902.08479
  - Existence/absence of **non-Hermitian skin effect** Kawabata=KS=Ueda=Sato 1812.09133, Kawabata=Okuma=Sato 2003.07597, ...
  - **Connection to quantum many-body physics**
  - **Experimental relevance**
  - And more...

Note: This is far from the exhaustive reference list on the topics above, due to the lack of my knowledge of recent developments.

38 Symmetry Classes in Finite Space Dimensions

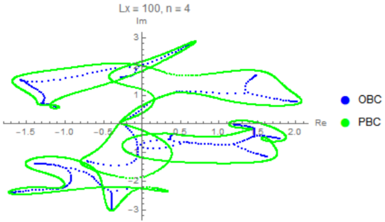
- In finite space dimensions (with $d \geq 1$), how we encode the 38 fundamental symmetries depends on the specific physical systems under consideration.
- One might focus on internal symmetries, which don't change the spatial position, as they remain compatible with the effects of the disorder.
- Here, we consider the following constraints on the hopping Hamiltonian $H(x, x')$:
  - Complex conjugation is local: $H(x, x')^* \leftrightarrow H(x, x')$.
  - Transpose exchanges the hopping direction: $H(x, x')^T \leftrightarrow H(x', x)$.

  This rule can be summarized in the table below:

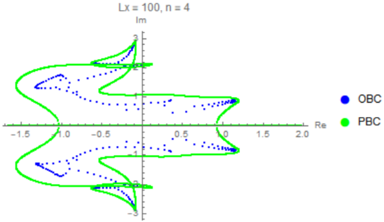
Symmetry	Symmetry in Real Space	With Translational Invariance
Unitary/SLS	$uH(x, x')u^\dagger = \pm H(x, x')$	$uH(k)u^\dagger = \pm H(k)$
TRS/PHS†	$uH(x, x')^* u^\dagger = \pm H(x, x')$	$uH(k)^* u^\dagger = \pm H(-k)$
TRS†/PHS	$uH(x, x')^T u^\dagger = \pm H(x', x)$	$uH(k)^T u^\dagger = \pm H(-k)$
PH/CS	$uH(x, x')^\dagger u^\dagger = \pm H(x', x)$	$uH(k)^\dagger u^\dagger = \pm H(k)$

A Numerical Experiment: PBC vs OBC for 38 symmetry classes

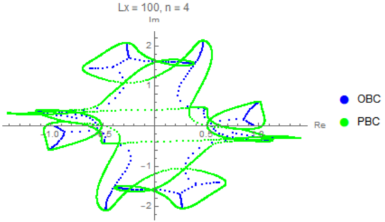
No symmetry



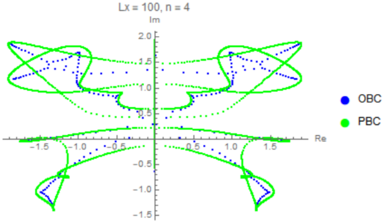
Pseudo Hermiticity $\sigma_z H(k)^\dagger \sigma_z = H(k)$



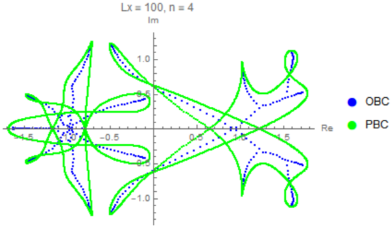
Sublattice symmetry $\sigma_z H(k) \sigma_z = -H(k)$



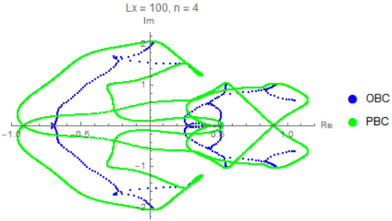
Chiral symmetry $\sigma_z H(k)^\dagger \sigma_z = -H(k)$



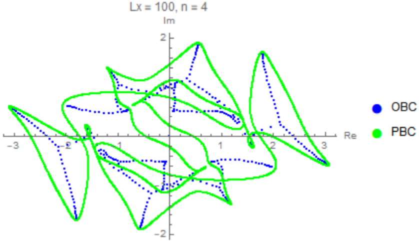
Class AI $\sigma_z H(k)^* \sigma_z = H(-k)$



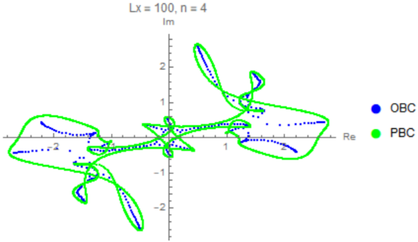
Class AII $\sigma_y H(k)^* \sigma_y = H(-k)$



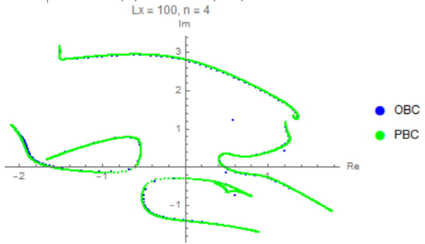
Class D $\sigma_z H(k)^T \sigma_z = -H(-k)$



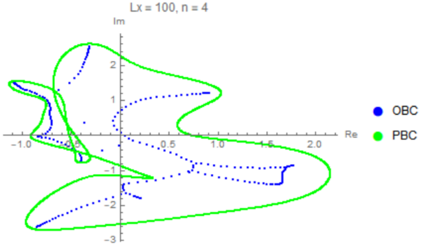
Class C $\sigma_y H(k)^T \sigma_y = -H(-k)$



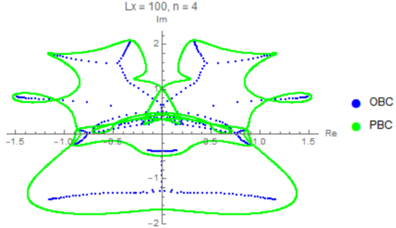
Class AI† $\sigma_z H(k)^T \sigma_z = H(-k)$



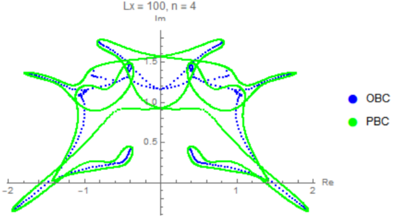
Class AII† $\sigma_y H(k)^T \sigma_y = H(-k)$



Class D† $\sigma_z H(k)^* \sigma_z = -H(-k)$



Class C† $\sigma_y H(k)^* \sigma_y = -H(-k)$



+ Other 28 classes → The PBC and OBC spectra are coincident if class AI† symmetry exists.

Kawabata–KS–Ueda–Sato 1812.09133, Kawabata–Okuma–Sato 2003.07597, ...

Topological Classification

Classification table of Hermitian topological phases "Periodic Table"

Schnyder=Ryu=Furusaki=Ludwig 0803.2786, Kitaev 0901.2686

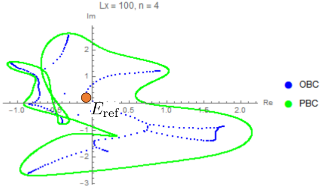
class\\δ	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+	+	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	−	+	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	−	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	−	−	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	−	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+	−	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

- Well-established. (The derivation is soon later.)

Point Gap and Hermitianization Gong=Ashida=Kawabata=Takasan=Higashikawa=Ueda 1802.07964

- The non-Hermitian skin effect is characterized by a nontrivial topological number with a point gap.

Class AII† $\sigma_y H(k)^T \sigma_y = H(-k)$



$$(-1)^\nu = \operatorname{sgn}\left[\frac{\operatorname{pf}[(H(0)-E_{\text{ref}})\sigma_y]}{\operatorname{pf}[(H(0)-E_{\text{ref}})\sigma_y]} \times \exp\frac{1}{2}\int_0^\pi d\log\det[(H(0)-E_{\text{ref}})\sigma_y]\right]$$

[Okuma=Kawabata=KS=Sato 1910.02878]

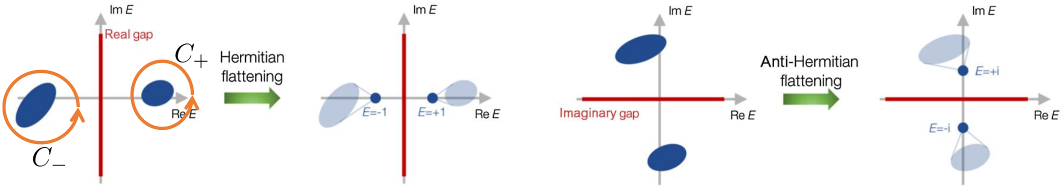
- How to systematically classify such topological phases/numbers? → Use the Hermitianization trick

$$\tilde{H}(k) = \begin{pmatrix} & H(k)^\dagger \\ H(k) & \end{pmatrix}.$$

- A point gap of $\tilde{H}(k)$ implies a gap of $\tilde{H}(k)$. This is because $\mathrm{Spec}(\tilde{H}(k)) = \mathrm{Spec}(\pm\sqrt{H(k)^\dagger H(k)})$.
- Classifying non-Hermitian $H(k)$ is recast as that of Hermitian Hamiltonian $\tilde{H}(k)$, which is well-established. → Done!

Line Gap and Flattening Kawabata=KS=Ueda=Sato 1812.09133

- With the real/imaginary line gap, non-Hermitian Hamiltonians H can be Hermite and flattened while keeping the real/imaginary line gap. $\to$ Done!



[Figure from Kawabata-KS-Ueda-Sato 1812.09133]

Proof (Based on App. D in Ashida=Gong=Ueda 2006.01837)

- For simplicity, from now on, we set $E_{\text{ref}} = 0$.

Flattening

- Let $C_+(C_-)$ be a circle enclosing all the eigenvalues with Re $E > 0$(Re $E < 0$).
- The projector onto the eigenspace with Re $E > 0$(Re $E < 0$) is given by

$$P_\pm(k) = \oint_{C_\pm} \frac{dz}{2\pi i} \frac{1}{z - H(k)}, \quad P_\pm(k)^2 = P_\pm(k).$$

- Introduce the homotopy

$$H_{t\in[0,1]}(k) = (1-t)H(k) + t[P_+(k) - P_-(k)],$$

whose eigenvalues are $(1-t)E_n(k) + t\,\mathrm{sgn}[\mathrm{Re}\, E_n(k)]$, which have a real line gap for $t \in [0,1]$.

- $H_1(k) = P_+(k) - P_-(k)$ has eigenvalues ± 1.

Hermitianization

- Decompose $H_1(k)$ into real and imaginary parts as

$$H_1(k) = h_1(k) + ih_2(k) = \frac{H_1(k) + H_1(k)^\dagger}{2} + i\frac{H_1(k) - H_1(k)^\dagger}{2i}.$$

- $H_1(k)^2 = P_+(k) + P_-(k) = 1$ implies that

$$h_1(k)^2 - h_2(k)^2 = 1, \quad \{h_1(k), h_2(k)\} = 0.$$

- Introduce the homotopy

$$\tilde{H}_{s\in[0,1]}(k) = (1-s)H_1(k) + sh_1(k) = h_1(k) + i(1-s)h_2(k),$$

  whose square is

$$\tilde{H}_s(k)^2 = h_1(k)^2 - (1-s)^2 h_2(k)^2 = 1 + (1-(1-s)^2)h_2(k)^2 \geq 1.$$

- Thus, $\tilde{H}_s(k)$ keeps the real line gap and $H_1(k)$ is Hermitianized to $h_1(k)$.
- $h_1(k)$ is not flat. We take the flattening to $h_1(k)$ again. □
- (Remark) These flattening and Hermitianization methods are compatible with 38 symmetries.

Topological Classification of Hermitian Systems

- For both point and line gaps, the classification problem is recast as that for Hermitian systems, which is well-established.

$$H(k)^\dagger = H(k), \quad H(k)^2 = 1 \ \text{(after flattening)}$$

- So, in the remainder of this section, I review the classification of Hermitian topological phases.
- Strategy: Classify 0-dimensional Hamiltonians and extend to finite space dimensions.

- (Remark) The classification of non-Hermitian topological phases here is for PBC. Due to the non-Hermitian skin effect, quantitative (and possibly qualitative) properties such as edge states must be discussed using the bulk Hamiltonian in OBC. The bulk-boundary correspondence is true between the bulk OBC Hamiltonian and the edge state. Yao=Wang 1803.01876, Yao=Song=Wang 1804.04672

Altland=Zirnbauer symmetry classes

- The fundamental internal symmetries are classified into 10-fold Altland-Zirnbauer (AZ) symmetry classes. Altland=Zirnbauer cond-mat/9602137
- There are three types of symmetries: [6]

$$\begin{aligned}
&\text{TRS:} && u_T H(x,x')^* u_T^\dagger = H(x,x') && u_T u_T^* = \pm 1, \\
&\text{PHS:} && u_C H(x,x')^* u_C^\dagger = -H(x,x') && u_C u_C^* = \pm 1, \\
&\text{Chiral:} && u_\Gamma H(x,x') u_\Gamma^\dagger = -H(x,x') && u_\Gamma^2 = 1, \quad \mathrm{tr}\,[u_\Gamma] = 0.
\end{aligned}$$

AZ class	TRS	PHS	Chiral
A	0	0	0
AIII	0	0	1
AI	1	0	0
BDI	1	1	1
D	0	1	0
DIII	−1	1	1
AII	−1	0	0
CII	−1	−1	1
C	0	−1	0
CI	1	−1	1

[6] $\mathrm{tr}\,[u_\Gamma] = 0$ is needed. Otherwise, H has zero modes.

Classifying Space

- We start with the classification of zero-dimensional Hamiltonian.

$$H^\dagger = H, \quad H^2 = 1 \; (\Leftrightarrow E = \pm 1) \quad + \text{AZ symmetry.}$$

- What is the "space" of such matrices?
- With "stable equivalence", such "spaces" become the *classifying spaces* in the K-theory. Kitaev 0901.2686

Example: 2×2 Hermitian matrix with $H^2 = 1$

- 2×2 Hermitian matrix H can be expanded as

$$H = d_0 + d_x \sigma_x + d_y \sigma_y + d_z \sigma_z = d_0 + \boldsymbol{d} \cdot \boldsymbol{\sigma}.$$

- Eigenvalues:

$$E = d_0 \pm |\boldsymbol{d}|.$$

- Thus, flattening implies either one of the following.
    - $d_0 = 1$ and $\boldsymbol{d} = \boldsymbol{0}$,
    - $d_0 = -1$ and $\boldsymbol{d} = \boldsymbol{0}$,
    - $d_0 = 0$ and $|\boldsymbol{d}| = 1$.
- Thus, there is a one-to-one correspondence

$$\{H \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) | H^\dagger = H, H^2 = 1\} = \underbrace{\{d_0 = 1\}}_{\mathrm{pt}} \cup \underbrace{\{\boldsymbol{d} \in S^2\}}_{\mathrm{Sphere}} \cup \underbrace{\{d_0 = -1\}}_{\mathrm{pt}}.$$

Stable equivalence Kitaev 0901.2686

- Practically, the homotopy classification of Hamiltonians whose target space is a finite and fixed dimension is not realistic.
- Even the classification is not a group.
- Example: class A 2×2 Hamiltonian in 3-space dimensions ("Hopf insulator Moore=Ran=Wen 0804.4527"):

$$[T^3, S^2] = \begin{cases} \text{(i) Three Chern numbers } (n_x, n_y, n_z) \in \mathbb{Z}^{\times 3} \\ \text{(ii) Hopf invariant is classified by } \mathbb{Z}_{2 \cdot \mathrm{GCD}(n_x, n_y, n_z)} \end{cases}$$

- The "stable equivalence condition" was introduced: Two Hamiltonians $H_0(k)$ and $H_1(k)$ are said stably equivalent $H_0(k) \sim H_1(k)$ if $H_0(k) \oplus H'(k)$ and $H_1(k) \oplus H'(k)$ are homotopically equivalent. [7]
- Physical motivation: stable against hybridization of higher- and lower-energy bands and the band folding by breaking translational symmetry.
- Mathematical motivation: (relatively) easy to compute.

[7] We further introduce the equivalence relation to pairs of Hamiltonians with the same size $(H_0(k), H_1(k))$. Two pairs $(H_0(k), H_1(k))$ and $(H_0'(k), H_1'(k))$ are equivalent if $H_0(k) \oplus H_1'(k) \sim H_0'(k) \oplus H_1(k)$. The equivalence classes form the K-theory.

Class A: Classifying Space C_0

- Let H be an $N \times N$ Hermitian matrix H with $H^2 = 1$.
- H is diagonalized by a unitary matrix

$$H = U \begin{pmatrix} 1_{N-M} & \\ & -1_M \end{pmatrix} U^\dagger,$$

where $M (0 \leq M \leq N)$ is the number of negative eigenvalues.

- U is not unique:

$$U \mapsto U \begin{pmatrix} V & \\ & W \end{pmatrix}, \quad V \in U(N-M), \quad W \in U(M).$$

- Thus, H is characterized by Grassmann manifolds

$$\bigcup_{M=0}^{N} \frac{U(N)}{U(N-M) \times U(M)}.$$

- With the stable equivalence [Kitaev 0901.2686], the Hamiltonian is eventually characterized by the classifying space C_0,

$$C_0 = \bigcup_{k \in \mathbb{Z}} \lim_{n \to \infty} \frac{U(2n)}{U(n+k) \times U(n-k)}.$$

Class AIII: Classifying Space C_1

- Let H be an $2N \times 2N$ Hermitian matrix H with $H^2 = 1$ and chiral symmetry

$$u_\Gamma H u_\Gamma^\dagger = -H, \quad u_\Gamma^2 = 1, \quad \mathrm{tr}\,[u_\Gamma] = 0.$$

- WLOG, we can set $u_\Gamma = \sigma_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. Then,

$$H = \begin{pmatrix} & q^\dagger \\ q & \end{pmatrix}, \quad q \in U(N).$$

- Thus, H is characterized by the unitary group $U(N)$.
- With the stable equivalence [Kitaev, 0901.2686], the Hamiltonian is eventually characterized by the classifying space C_1,

$$C_1 = \lim_{n \to \infty} U(n).$$

Class AIII: Classifying Space C_1 (alternative)

- There is another perspective on C_1.
- Start with the diagonalization $H = U\sigma_z U^\dagger$.
- Set $u_\Gamma = \sigma_x$. The symmetry $\sigma_x H \sigma_x = -H$ implies that one can choose $\sigma_x U = U\sigma_x$. Namely,

$$U = u_+ P_+ + u_- P_- = \frac{1}{2}\begin{pmatrix} u_+ + u_- & u_+ - u_- \\ u_+ - u_- & u_+ + u_- \end{pmatrix}, \quad u_+, u_- \in U(N).$$

  where $P_\pm = \frac{1 \pm \sigma_x}{2}$ is the projection onto $\sigma_x = \pm 1$.
- The redundancy of U is $U \mapsto UV$ with $V\sigma_z V^\dagger = \sigma_z$ and $\sigma_x V = V\sigma_x$. Thus, V is a form $V = \sigma_y \otimes \tilde{V}, \tilde{V} \in U(N)$.
- We got

$$C_1 = \lim_{n\to\infty} [U(n) \times U(n)]/U(n).$$

Class AI: Classifying Space R_0

- Let H be an $N \times N$ Hermitian matrix H with $H^2 = 1$ and class AI TRS

$$u_T H^* u_T^\dagger = H, \quad u_T u_T^* = 1.$$

- WLOG, we can set $u_T = 1$ [8], meaning that H is diagonalized by an orthogonal matrix

$$H = O \begin{pmatrix} 1_{N-M} & \\ & -1_M \end{pmatrix} O^T.$$

- The same logic as class A leads the classifying space R_0,

$$R_0 = \bigcup_{k \in \mathbb{Z}} \lim_{n \to \infty} \frac{O(2n)}{O(n+k) \times O(n-k)}.$$

[8] Every symmetric matrix $u_T^T = u_T$ can be $u_T = Q \Lambda Q^T$ with $\Lambda \geq 0$ and Q a unitary (Autonne–Takagi factorization). When u_T is unitary, $\Lambda = 1$.

Class BDI: Classifying Space R_1

- Let H be an $N \times N$ Hermitian matrix H with $H^2 = 1$ and class BDI symmetry

$$\begin{aligned} &u_T H^* u_T^\dagger = H, \quad u_T u_T^* = 1, \\ &u_\Gamma H u_\Gamma^\dagger = -H, \quad u_\Gamma^2 = 1, \quad \mathrm{tr}\,[u_\Gamma] = 0, \\ &u_T u_\Gamma^* = u_\Gamma u_T. \end{aligned}$$

- We can set $u_\Gamma = \sigma_z$ and $u_T = 1$, meaning that q is an orthogonal matrix

$$H = \begin{pmatrix} & q^\dagger \\ q & \end{pmatrix}, \quad q \in O(N).$$

- We get the classifying space R_1,

$$R_1 = \lim_{n\to\infty} O(n).$$

- The $\mathbb{Z}_2$ invariant is given by $\det q \in \{\pm 1\}$.
- As for C_1, it can also be obtained as $R_1 = \lim_{n\to\infty} [O(n) \times O(n)]/O(n)$.

Class D: Classifying Space R_2

- Let H be an $2N \times 2N$ Hermitian matrix H with $H^2 = 1$ and class D PHS

$$u_C H^* u_C^\dagger = -H, \quad u_C u_C^* = 1.$$

- We can set $u_C = 1$, meaning that iH is a real skew-symmetric matrix, which is diagonalized as

$$iH = O \left[1_N \otimes \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \right] O^T, \quad O \in O(2n).$$

- O is not unique:

$$O \mapsto O \begin{pmatrix} \text{Re } U & \text{Im } U \\ -\text{Im } U & \text{Re } U \end{pmatrix}, \quad \text{Re } U = \frac{U + U^*}{2}, \quad \text{Im } U = \frac{U - U^*}{2i}, \quad U \in U(n).$$

$$\to R_2 = \lim_{n \to \infty} \frac{O(2n)}{U(n)}.$$

- The $\mathbb{Z}_2$ invariant is given by $\mathrm{pf}\,[iH] = \det O \in \{\pm 1\}$.

Class D: Classifying Space R_2 (alternative)

- Start with the diagonalization $H = U\sigma_z U^\dagger$.
- Set $u_C = 1$. Then, the symmetry constraint $H^* = -H$ implies that U can be chosen as $U^* = U\sigma_x$, which is the same as $V = Ue^{\frac{i\pi}{4}(\sigma_x - 1)}$ is real $V^* = V$.
- Then, $H = V(-\sigma_y)V^\dagger$.
- The redundancy of V is $V \mapsto VQ$ with $Q^* = Q$ and $Q\sigma_y Q^\dagger = \sigma_y$, which means $Q \in U(N)$ as before.
- We get

$$R_2 = \lim_{n\to\infty} \frac{O(2n)}{U(n)}.$$

Class DIII: Classifying Space R_3

- Let H be an $4N \times 4N$ Hermitian matrix H with $H^2 = 1$ and class CI symmetry

$$u_T H^* u_T^\dagger = H, \quad u_T u_T^* = 1,$$
$$u_\Gamma H u_\Gamma^\dagger = -H, \quad u_\Gamma^2 = 1, \quad \mathrm{tr}\,[u_\Gamma] = 0,$$
$$u_T u_\Gamma^* = -u_\Gamma u_T.$$

- We can set $u_\Gamma = \sigma_z$ and $u_T = \sigma_x \tau_y$. Then, the symmetry constraint is recast as follows.

$$H = \begin{pmatrix} & q^\dagger \\ q & \end{pmatrix}, \quad \tau_y q^T \tau_y = q.$$

- The matrix $\tau_y q$ is a complex skew-symmetric and unitary, meaning that it can be a form $\tau_y q = Q(i\sigma_y)Q^T$ with $Q \in U(2N)$.
- The redundancy of Q is $Q \mapsto QV$ with $VV^\dagger = 1$ and $V(i\sigma_y)V^T = i\sigma_y$. Namely, $V \in Sp(N)$.
- We get

$$R_3 = \lim_{n\to\infty} \frac{U(2n)}{Sp(n)}.$$

Class AII: Classifying Space R_4

- Let H be an $2N \times 2N$ Hermitian matrix H with $H^2 = 1$ and class AII TRS

$$u_T H^* u_T^\dagger = H, \quad u_T u_T^* = -1.$$

- We can set $u_T = i\sigma_y = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ [9]. The eigenvectors come in Kramers pairs

$$(u_{2i-1}, u_{2i}) = (u_{2i-1}, i\sigma_y u_{2i-1}^*),$$

meaning that H is diagonalized by a compact symplectic matrix

$$H = S \begin{pmatrix} 1_{N-M} & \\ & -1_M \end{pmatrix} S^\dagger, \quad S \in Sp(N) = Sp(2N;\mathbb{C}) \cap U(2N) = \{S \in U(2N) | S^T i\sigma_y S = i\sigma_y\}.$$

$$\to R_4 = \bigcup_{k\in\mathbb{Z}} \lim_{n\to\infty} \frac{Sp(2n)}{Sp(n+k)\times Sp(k-n)}.$$

[9] Every skew-symmetric matrix $u_T^T = -u_T$ can be $u_T = Q\Lambda Q^T$ with $\Lambda = \bigoplus_i \begin{pmatrix} & \lambda_i \\ -\lambda_i & \end{pmatrix} Q^T$ with Q a unitary. When u_T is unitary, λ_is can be $\lambda_i \equiv 1$.

Class CII: Classifying Space R_5

- Let H be an $N \times N$ Hermitian matrix H with $H^2 = 1$ and class CII symmetry

$$u_T H^* u_T^\dagger = H, \quad u_T u_T^* = -1,$$
$$u_\Gamma H u_\Gamma^\dagger = -H, \quad u_\Gamma^2 = 1, \quad \mathrm{tr}\left[u_\Gamma\right] = 0,$$
$$u_T u_\Gamma^* = u_\Gamma u_T.$$

- We can set $u_\Gamma = \sigma_z$ and $u_T = \tau_y$. Then, the symmetry constraint is recast as follows.

$$H = \begin{pmatrix} & q^\dagger \\ q & \end{pmatrix}, \quad \tau_y q^* \tau_y = q \Leftrightarrow q \tau_y q^T = \tau_y.$$

- We get

$$R_5 = \lim_{n\to\infty} Sp(n).$$

- As for C_1, it can be obtained as $R_5 = \lim_{n\to\infty} [Sp(n) \times Sp(n)]/Sp(n)$.

Class C: Classifying Space R_6

- Start with the diagonalization $H = U\sigma_z U^\dagger$.
- Set $u_C = \sigma_y$. Then, the symmetry constraint $\sigma_y H^* = -H\sigma_y$ implies that U can be chosen as $\sigma_y U^* = U\sigma_y$. Namely, $U \in Sp(N)$.
- The redundancy of U is $U \mapsto UV$ with $V\sigma_y V^T = \sigma_y$ and $V\sigma_z V^\dagger = \sigma_z$, which means $V = \begin{pmatrix} v & \\ & v^* \end{pmatrix}$ with $v \in U(N)$.
- We get

$$R_6 = \lim_{n\to\infty} \frac{Sp(n)}{U(n)}.$$

Class CI: Classifying Space R_7

- Let H be an $2N \times 2N$ Hermitian matrix H with $H^2 = 1$ and class CI symmetry

$$\begin{aligned} &u_T H^* u_T^\dagger = H, \quad u_T u_T^* = 1, \\ &u_\Gamma H u_\Gamma^\dagger = -H, \quad u_\Gamma^2 = 1, \quad \mathrm{tr}\,[u_\Gamma] = 0, \\ &u_T u_\Gamma^* = -u_\Gamma u_T. \end{aligned}$$

- We can set $u_\Gamma = \sigma_z$ and $u_T = \sigma_x$. Then, the symmetry constraint is recast as follows.

$$H = \begin{pmatrix} & q^\dagger \\ q & \end{pmatrix}, \quad q^T = q.$$

- The complex symmetric and unitary matrix can be a form $q = QQ^T$ with $Q \in U(N)$.
- The redundancy of Q is $Q \mapsto QV$ with $VV^\dagger = 1$ and $VV^T = 1$. Namely, $V \in O(N)$.
- We get

$$R_7 = \lim_{n\to\infty} \frac{U(n)}{O(n)}.$$

Classifying Space

- Eventually, we get the 10 classifying spaces and their disconnected parts. [10]

AZ class	TRS	PHS	Chiral	Classifying Space	π_0	Top. invariant
A	0	0	0	$C_0 = \cup_{k\in\mathbb{Z}} \lim_{n\to\infty} \frac{U(2n)}{U(n+k)\times U(n-k)}$	$\mathbb{Z}$	$k \in \mathbb{Z}$
AIII	0	0	1	$C_1 = \lim_{n\to\infty} U(n)$	0	
AI	1	0	0	$R_0 = \cup_{k\in\mathbb{Z}} \lim_{n\to\infty} \frac{O(2n)}{O(n+k)\times O(n-k)}$	$\mathbb{Z}$	$k \in \mathbb{Z}$
BDI	1	1	1	$R_1 = \lim_{n\to\infty} O(n)$	$\mathbb{Z}_2$	$\det q \in \pm 1$
D	0	1	0	$R_2 = \lim_{n\to\infty} \frac{O(2n)}{U(n)}$	$\mathbb{Z}_2$	$\mathrm{pf}\,[iH] \in \pm 1$
DIII	-1	1	1	$R_3 = \lim_{n\to\infty} \frac{U(2n)}{Sp(n)}$	0	
AII	-1	0	0	$R_4 = \cup_{k\in\mathbb{Z}} \lim_{n\to\infty} \frac{Sp(2n)}{Sp(n+k)\times Sp(n-k)}$	$2\mathbb{Z}$	$k \in \mathbb{Z}$
CII	-1	-1	1	$R_5 = \lim_{n\to\infty} Sp(n)$	0	
C	0	-1	0	$R_6 = \lim_{n\to\infty} \frac{Sp(n)}{U(n)}$	0	
CI	1	-1	1	$R_7 = \lim_{n\to\infty} \frac{U(n)}{O(n)}$	0	

[10] $Sp(N) = Sp(2N;\mathbb{C}) \cap U(2N) = \{S \in U(2N) | S^T i\sigma_y S = i\sigma_y\}$

Finite Space Dimensions (i) from torus to sphere

- Thanks to the stable equivalence, the topological structure from "different origins" can be discussed independently.
- For d-spatial dimensions, the Bloch-momentum space is a d-dimensional torus T^d, however, with stable equivalence, the topological classification is decomposed into that of sub-spheres S^p, $0 \leq p \leq d$, like

$$\text{“}H(\text{Skyrmion} + \text{Vortex})\text{”} \rightarrow \text{“}H(\text{Skyrmion}) \oplus H(\text{Vortex})\text{”}.$$

- We can assume the Bloch-momentum space is a d-sphere.

Finite Space Dimensions (ii) Dirac Hamiltonians

- Moreover, it is found that the representative Hamiltonian can be a form of the Dirac Hamiltonian

$$H(k) = \sum_{i=1}^{d} k_i \gamma_i + M, \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij}, \quad \{\gamma_i, M\} = 0, \quad M^2 = 1.$$

- The topological classification of $H(k)$ is recast as the classification of the mass term M subject to the constraint by γ_is and AZ symmetry.
- Adding space dimensions $d = 1, 2, \ldots$ is the same as adding gamma matrices $\gamma_1, \gamma_2, \ldots$.
- The gamma matrices γ_is behave as chiral symmetries.

Dimensional isomorphism

- We will show that adding gamma matrices is nothing but a shift of AZ symmetry class.

$$\cdots \to \mathrm{A} \to \mathrm{AIII} \to \mathrm{A} \to \cdots \quad \text{(without TRS and PHS)},$$

$$\cdots \mathrm{AI} \to \mathrm{CI} \to \mathrm{C} \to \mathrm{CII} \to \mathrm{AII} \to \mathrm{DIII} \to \mathrm{D} \to \mathrm{BDI} \to \mathrm{AI} \to \cdots .$$

- The key observation is that two chiral symmetries can be "solved" trivially:

$$\{\sigma_x, M\} = \{\sigma_y, M\} = 0 \quad \Rightarrow \quad M = \sigma_z \otimes \tilde{M}.$$

A → AIII → A

- Let us consider a $d = 1$ class A Dirac Hamiltonian

$$H(k_1) = k_1\gamma_1 + M, \quad \{\gamma_1, M\} = 0.$$

- γ_1 behaves as chiral symmetry, thus,

$$(d = 1,\ \text{class A}) \quad = \quad (d = 0,\ \text{class AIII}).$$

- Next, let us consider a $d = 1$ class AIII Dirac Hamiltonian

$$H(k_1) = k_1\gamma_2 + M, \quad \{\gamma_2, M\} = 0,$$
$$\gamma_1 H(k_1)\gamma_1^\dagger = -H(k_1).$$

- We can set $\gamma_1 = \sigma_x$ and $\gamma_2 = \sigma_z$. Then,

$$M = \sigma_y \otimes \tilde{M}.$$

- No constraints on $\tilde{M}$ exist, meaning that

$$(d = 1,\ \text{class AIII}) \quad = \quad (d = 0,\ \text{class A}).$$

Dimensional isomorphism with TRS or PHS

- With antiunitary symmetry, we chase the change of AZ symmetry for $\tilde{M}$.
- The symmetry constraint

$$u_T H(k)^* u_T^\dagger = H(-k),$$
$$u_T H(k)^* u_T^\dagger = -H(-k)$$

implies that

$$u_T \gamma_i^* u_T^\dagger = -\gamma_i, \quad u_T M^* u_T^\dagger = M,$$
$$u_C \gamma_i^* u_C^\dagger = \gamma_i, \quad u_C M^* u_C^\dagger = -M.$$

AI $\to$ CI

- Let us consider a $d = 1$ class AI Dirac Hamiltonian

$$H(k_1) = k_1\gamma_1 + M, \quad \{\gamma_1, M\} = 0.$$

The symmetry algebra

$$u_T\gamma_1^* u_T^\dagger = -\gamma_1, \quad u_T u_T^* = 1,$$

is solved by

$$u_T = \sigma_x, \quad \gamma_1 = \sigma_z.$$

Introducing PHS $u_C = i\gamma_1 u_T = \sigma_y$, the constraint on the matrix M is the same as class CI:

$$u_T M^* u_T^\dagger = M, \quad u_T u_T^* = 1,$$
$$u_C M^* u_C^\dagger = -M, \quad u_C u_C^* = -1.$$

Thus,

$$(d = 1, \text{ class AI}) \quad = \quad (d = 0, \text{ class CI}).$$

CI → C

- Let us consider a $d = 1$ class CI Dirac Hamiltonian

$$H(k_1) = k_1\gamma_1 + M,$$
$$u_C\gamma_1^* u_C^\dagger = \gamma_1, \quad u_C M^* u_C^\dagger = -M, \quad u_C u_C^* = -1,$$
$$u_\Gamma\gamma_1 u_\Gamma^\dagger = -\gamma_1, \quad u_\Gamma M u_\Gamma^\dagger = -M, \quad u_\Gamma^2 = 1,$$
$$u_C u_\Gamma^* = -u_\Gamma u_C.$$

- We can set u_Γ, γ_1, and M as

$$u_\Gamma = \sigma_x, \quad \gamma_1 = \sigma_z, \quad M = \sigma_y \otimes \tilde{M}.$$

- The only remaining symmetry is u_c, which should be a form

$$u_C = \sigma_z \otimes \tilde{u}_C, \quad \tilde{u}_C\tilde{u}_C^* = -1,$$

and constrain the mass term $\tilde{M}$ as

$$\tilde{u}_C\tilde{M}^*\tilde{u}_C^\dagger = -\tilde{M}.$$

- Thus,

$$(d = 1, \text{ class CI}) \quad = \quad (d = 0, \text{ class C}).$$

Dimensional isomorphism

- In this way, we have the shift of AZ symmetry classes by adding space dimensions

$$\cdots \to \mathrm{A} \to \mathrm{AIII} \to \mathrm{A} \to \cdots \quad \text{(without TRS and PHS)},$$

$$\cdots \mathrm{AI} \to \mathrm{CI} \to \mathrm{C} \to \mathrm{CII} \to \mathrm{AII} \to \mathrm{DIII} \to \mathrm{D} \to \mathrm{BDI} \to \mathrm{AI} \to \cdots .$$

- These also show the Bott periodicity

$$C_{n-2} = C_n, \quad R_{n-8} = R_n.$$

- Eventually, the topological classification of d-dimensional Hamiltonian $H(k)$ with AZ symmetry C_n or R_n is given by

$$\pi_0[C_{n-d}] \quad \text{and} \quad \pi_0[R_{n-d}].$$

$\to$ periodic table.

Identify Mapped Symmetry

- The remaining task is to identify how 38 non-Hermitian symmetry classes are mapped to 10 AZ Hermitian symmetry classes for each gap condition.
- For the point gap, the Hermitianized doubled Hamiltonian

$$\tilde{H}(k) = \begin{pmatrix} & H(k)^\dagger \\ H(k) & \end{pmatrix}$$

has additional chiral symmetry

$$\sigma_z \tilde{H}(k) \sigma_z = -\tilde{H}(k).$$

Other internal symmetries are mapped for a symmetry constraint of $\tilde{H}(k)$ and commutation/anticommutation relation with σ_z.

- For the real (imaginary) line gap, $H(k)$ can be (anti-)Hermite $H(k)^\dagger = H(k)$ $(H(k)^\dagger = -H(k))$. The (anti-)Hermitian condition of $H(k)$ is the same as imposing an additional chiral symmetry on $\tilde{H}(k)$:

$$\sigma_y \tilde{H}(k) \sigma_y = -\tilde{H}(k) \quad \text{for real line gap,}$$
$$\sigma_x \tilde{H}(k) \sigma_x = -\tilde{H}(k) \quad \text{for imaginary line gap.}$$

Other internal symmetries have definite commutation/anticommutation relations with σ_y (σ_x).

Classification tables of non-Hermitian topological phases Kawabata=KS=Ueda=Sato arXiv:1812.09133, cf. Gong=Ashida=Kawabata=Takasan=Higashikawa=Ueda 1802.07964, Zhou=Lee 1812.10490

AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
AI	P	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
	$\mathrm{L_r}$	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	$\mathrm{L_i}$	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
BDI	P	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
	$\mathrm{L_r}$	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
	$\mathrm{L_i}$	$\mathcal{R}_2 \times \mathcal{R}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}$	0	0	0	$2\mathbb{Z} \oplus 2\mathbb{Z}$	0
D	P	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
	L	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	P	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	$\mathrm{L_r}$	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
	$\mathrm{L_i}$	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AII	P	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	$\mathrm{L_r}$	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	$\mathrm{L_i}$	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CII	P	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	$\mathrm{L_r}$	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	$\mathrm{L_i}$	$\mathcal{R}_6 \times \mathcal{R}_6$	0	0	$2\mathbb{Z} \oplus 2\mathbb{Z}$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}$	0
C	P	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	L	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	P	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	$\mathrm{L_r}$	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	$\mathrm{L_i}$	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

+ 30 other symmetry classes. (See Kawabata=KS=Ueda=Sato arXiv:1812.09133 for the details.)

Intrinsic Non-Hermitian Topology

AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
AI	P	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
	$\mathrm{L_r}$	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	$\mathrm{L_i}$	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
BDI	P	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	$\mathrm{L_r}$	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	$\mathrm{L_i}$	$\mathcal{R}_2 \times \mathcal{R}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}$	0	0	0	$2\mathbb{Z} \oplus 2\mathbb{Z}$	0
D	P	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
	L	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	P	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	$\mathrm{L_r}$	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	$\mathrm{L_i}$	$\mathcal{C}_0$	$\mathbb{Z}$	0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
AII	P	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	$\mathrm{L_r}$	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	$\mathrm{L_i}$	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CII	P	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	$\mathrm{L_r}$	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	$\mathrm{L_i}$	$\mathcal{R}_6 \times \mathcal{R}_6$	0	0	$2\mathbb{Z} \oplus 2\mathbb{Z}$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}$	0
C	P	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	L	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	P	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	$\mathrm{L_r}$	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	$\mathrm{L_i}$	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

?

Edge Majorana zero mode

Motivating example: 1d class D non-Hermitian superconductor

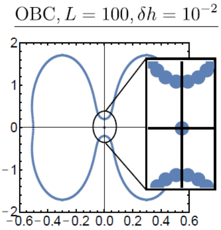
- Class D PHS symmetry:

$$\tau_x H(k_x)^T \tau_x = -H(-k_x), \quad E \to -E.$$

- Both the point gap and line gap show the $\mathbb{Z}_2$ classification.
- Non-Hermitian $\mathbb{Z}_2$ invariant:

$$(-1)^\nu = \operatorname{sgn}\left\{ \frac{\mathrm{Pf}[H(\pi)\tau_x]}{\mathrm{Pf}[H(0)\tau_x]} \times \exp\left[-\frac{1}{2} \int_0^\pi d \log \det [H(k)\tau_x] \right] \right\}$$

- If $(-1)^\nu = -1$, there is a Majorana zero mode at each edge Kawabata=KS=Ueda=Sato 1812.09133.



[Figure from Okuma=Sato 1904.06355]

- Unique to non-Hermitian systems?

Topological phenomena unique to non-Hermitian systems

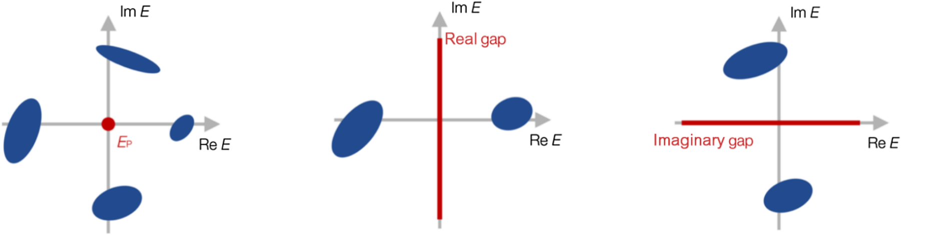
- Sometimes, we encounter topological phases which are realized only in non-Hermitian systems.
- On the other hand, there are topological phases that are remnant in non-Hermitian systems. For instance, the Chern insulator with a small non-Hermite perturbation is still characterized by the Chern number of the Bloch wave function.

- Is there any good approach to extracting topological phases realized only in the presence of non-Hermiticity?
- Our proposal [Sec.IX in Supplemental Material of Okuma=Kawabata=KS=Sato 1910.02878]: Take the cokernel of the following map

$$\text{Line-gapped topological phases} \quad \longrightarrow \quad \text{Point-gapped topological phases}$$

Line gap $\Rightarrow$ point gap

- If a line gap is open, the point gap is also open.



[Figure from Kawabata=KS=Ueda=Sato 1812.09133]

- This implies that there exist homomorphisms f_{r} and f_{i} from the real and imaginary line-gapped topological phases to the point-gapped topological phases!

$$f_{\mathrm{r}} : \text{(Real line-gapped topological phases)} \to \text{(Point-gapped topological phases)},$$
$$f_{\mathrm{i}} : \text{(Imaginary line-gapped topological phases)} \to \text{(Point-gapped topological phases)}.$$

Intrinsic non-Hermitian Topology

- The point-gapped topological phases that are in the image

$$\text{Im}\ f_{\mathrm{r}} + \text{Im}\ f_{\mathrm{i}} \subset (\text{Point-gapped topological phases})$$

can be deformed into a real or imaginary line-gapped topological phase while keeping the point gap.

- Such point-gapped topological phases are also realized in Hermitian or anti-Hermitian systems.
- Importantly, their physics such as the bulk-boundary correspondence can be understood in Hermitian or anti-Hermitian systems.
- On the other hand, the quotient

$$(\text{Point-gapped topological phases})/(\text{Im}\ f_r + \text{Im}\ f_i)$$

represents topological phases intrinsic to non-Hermitian systems.

- Thanks to the dimensional isomorphism introduced before, it suffices to calculate the homomorphisms $f_{\mathrm{r}}, f_{\mathrm{i}}$ from line-gapped to point-gapped topological phases only for $d = 0$.

Results: AZ class

Tables from Okuma=Kawabata=KS=Sato 1910.02878.

AZ class	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	0	0	0	0	0	0
AI	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	0
BDI	0	0	0	0	0	0	0	0
D	0	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
DIII	0	0	0	0	$\mathbb{Z}_2$	0	0	0
AII	0	$2\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	0
CII	0	0	0	0	0	0	0	0
C	0	0	0	$2\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CI	$\mathbb{Z}_2$	0	0	0	0	0	0	0

- $d=1$, class A: non-Hermitian skin effect.
- $d=3$, class A: non-Hermitian skin effect induced by a magnetic field. Bessho=Sato 2006.04204, Kawabata=Shiozaki=Ryu 2011.11449

AZ† class

AZ† class	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
AI†	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI†	0	0	0	0	0	0	0	0
D†	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	0
DIII†	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0	0	0
AII†	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
CII†	0	0	0	0	0	0	0	0
C†	0	$2\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	0
CI†	0	0	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0

- $d=1,2$, class AII†: $\mathbb{Z}_2$ non-Hermitian skin effect. Okuma=Kawabata=KS=Sato 1910.02878

AZ class with sublattice symmetry or pseudo-Hermiticity

AZ class	Add. symm.	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
A	η	0	0	0	0	0	0	0	0
AIII	S_+, η_+	0	0	0	0	0	0	0	0
A	S	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	S_-, η_-	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
AI	η_+	0	0	0	0	0	0	0	0
BDI	S_{++}, η_{++}	0	0	0	0	0	0	0	0
D	η_+	0	0	0	0	0	0	0	0
DIII	S_{--}, η_{++}	0	0	0	0	0	0	0	0
AII	η_+	0	0	0	0	0	0	0	0
CII	S_{++}, η_{++}	0	0	0	0	0	0	0	0
C	η_+	0	0	0	0	0	0	0	0
CI	S_{--}, η_{++}	0	0	0	0	0	0	0	0

- $d=2$, class AII$+S_-$: Edge exceptional point Denner–Neupert–Schindler 2304.13743

(cont.)

AZ class	Add. symm.	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
AI	S_-	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	0
BDI	S_{-+}, η_{+-}	0	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
D	S_+	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$
DIII	S_{-+}, η_{-+}	0	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
AII	S_-	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	0
CII	S_{-+}, η_{+-}	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0	0	0
C	S_+	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CI	S_{-+}, η_{-+}	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0	0	0
AI	η_-	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0	0	0
BDI	S_{--}, η_{--}	0	0	0	0	0	0	0	0
D	η_-	0	0	0	0	$\mathbb{Z}_2$	0	0	0
DIII	S_{++}, η_{--}	0	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0
AII	η_-	0	0	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
CII	S_{--}, η_{--}	0	0	0	0	0	0	0	0
C	η_-	$\mathbb{Z}_2$	0	0	0	0	0	0	0
CI	S_{++}, η_{--}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0	0	0	0

(cont.)

AZ class	Add. symm.	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
AI	S_+	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
BDI	S_{+-}, η_{-+}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}_2$	0
D	S_-	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
DIII	S_{+-}, η_{+-}	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0
AII	S_+	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0
CII	$S_{+-}, \eta -+$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
C	S_-	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI	S_{+-}, η_{+-}	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

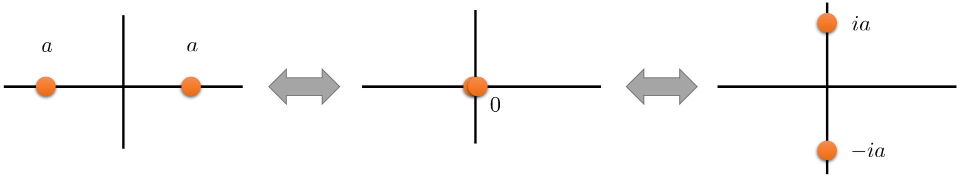
Note: I'm not familiar with the current status of the studies of intrinsic non-Hermitian topological phases. The reference list above may be very limited.

Example: Class AIII+S$_-$ (sublattice symmetry anti-commuting with chiral symmetry)

- Symmetry:

$$\begin{cases} \sigma_z H(\boldsymbol{k}) \sigma_z = -H(\boldsymbol{k}), \\ \sigma_y H(\boldsymbol{k})^\dagger \sigma_y = -H(\boldsymbol{k}). \end{cases} \quad \Rightarrow \quad H(\boldsymbol{k}) = \begin{pmatrix} & h_1(\boldsymbol{k}) \\ h_2(\boldsymbol{k}) & \end{pmatrix}, \qquad h_j(\boldsymbol{k})^\dagger = h_j(\boldsymbol{k}) \quad (j = 1, 2).$$

- $d = 0$: (Point-gapped topological phases)/(Im $f_r \cup$ Im f_i) $= \mathbb{Z}_2$.
  $\rightarrow$ is understood as the existence of the PT-symmetry breaking accompanied with an exceptional point at $E = 0$:



$$H = \begin{bmatrix} & a \\ a & \end{bmatrix} \qquad H = \begin{bmatrix} & a \\ 0 & \end{bmatrix} \qquad H = \begin{bmatrix} & a \\ -a & \end{bmatrix}$$

Example: Class AIII+S$_-$ (cont.)

- $d=2$: (Point-gapped topological phases)/(Im $f_r \cup$ Im f_i) $= \mathbb{Z}_2$.
- There exists an intrinsic non-Hermitian topological phase.
- A model:

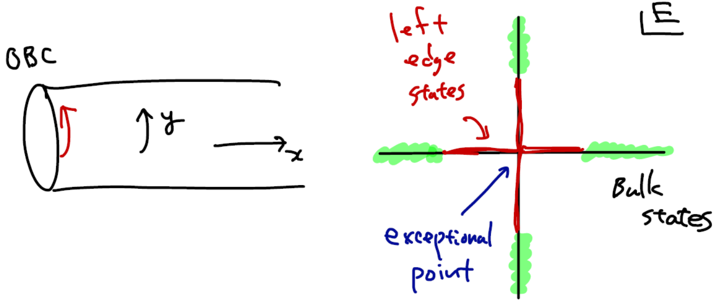
$$H(k_x, k_y) = \begin{pmatrix} & h_{\text{Chern}}(k_x, k_y) \\ \mathbf{1}_{2\times 2} & \end{pmatrix},$$

$$h_{\text{Chern}}(k_x, k_y) = \sin k_x \sigma_x + \sin k_y \sigma_y + (m - t\cos k_x - t\cos k_y)\sigma_z.$$

- $H = \begin{pmatrix} & \epsilon \\ 1 & \end{pmatrix} \quad \Rightarrow \quad \begin{cases} E = \pm\sqrt{\epsilon} & (\epsilon > 0) \\ E = \pm i\sqrt{-\epsilon} & (\epsilon < 0) \end{cases}$

Example: Class AIII+S_- (cont.)

- The Chern insulator $h_{\mathrm{Chern}}(k_x, k_y)$ has a chiral edge state localized at each edge.
- Therefore, the non-Hermitian Hamiltonian $H(k_x, k_y)$ has an exceptional point, the trajectory of the "PT-symmetry breaking", at each edge. Denner=Neupert=Schindler 2304.13743



Summary

In this lecture, I gave

1. Introduction
   — One-particle non-Hermitian systems
   — Exceptional point
   — Non-Hermitian skin effect
2. Gap condition and topology
   — Point gap
   — Real and imaginary line gaps
3. Symmetry classes
   — 38 classes in non-Hermitian systems
4. Topological classification
   — Point gap → doubled Hermitian Hamiltonian → Hermitian topological phases
   — Line gap → Hermitianization → Hermitian topological phases
   — Classifying spaces
   — Dimensional isomorphism
5. Intrinsic non-Hermitian topology
   — Line gap implies point gap
   — Intrinsic non-Hermitian topological phases should be interesting!

Skin effect is topological Zhang-Yang-Fang 19, Okuma-Kawabata-KS-Sato 19

- $W(H(k)) := \frac{1}{2\pi i}\oint d\log\det[H_{\mathrm{PBC}}(k)] \neq 0 \Rightarrow$ skin effect.

(Our proof)

- Let $\sigma(H_{\mathrm{PBC}})$, $\sigma(H_{\mathrm{OBC}})$ and $\sigma(H_{\mathrm{SIBC}})$ be the spectrum for PBC, OBC and the semi-infinite bdy condition, respectively. It holds that

$$\sigma(H_{\mathrm{OBC}}) \subset \sigma(H_{\mathrm{SIBC}}).$$

- The spectrum for OBC is invariant under the similarity transformation

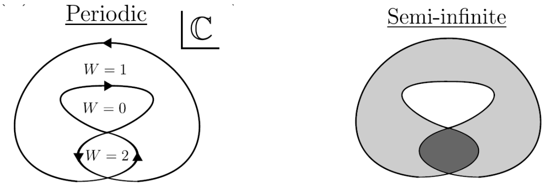
$$V_g f_x^\dagger V_g^\dagger = e^g f_x^\dagger, \quad g \in (0,\infty).$$

Therefore,

$$\sigma(H_{\mathrm{OBC}}) \subset \bigcap_{g\in(-\infty,\infty)} \sigma(V_g^{-1} H_{\mathrm{SIBC}} V_g).$$

- Toeplitz index theorem:

$$\sigma(H_{\mathrm{SIBC}}) = \sigma(H_{\mathrm{PBC}}) \cup \underbrace{\{E \in \mathbb{C} | W(H(k) - E) \neq 0\}}_{\text{dense spectrum}}.$$



This is because the bulk-boundary correspondence for the class AIII doubled Hamiltonian

$$\tilde{H}(k) = \begin{pmatrix} & H(k) - E \\ H(k)^\dagger - E^* & \end{pmatrix}.$$

If $W(H(k) - E) < 0$, there exists a zero mode $(0, |E\rangle)^T$ of $\tilde{H}$, i.e., the right eigenstate of $H(k)$ with eigenvalue E.

Skin effect is topological (cont.) Zhang-Yang-Fang 19, Okuma-Kawabata-KS-Sato 19

- Suppose that $H_{\mathrm{PBC}}(k)$ has a nonzero winding number.
- Take an arbitrary complex energy E with $W(H_{\mathrm{PBC}}(k)-E)\neq 0$. $|E\rangle$ represents an right or left eigenstate localized at the boundary.
- There exists $g\in(0,\infty)$ s.t. $|E\rangle$ such that $|E\rangle$ is a delocalized plane wave of $V_g^{-1}H_{\mathrm{SIBC}}V_g$, i.e. $E\in\sigma(V_g^{-1}H_{\mathrm{PBC}}V_g)$.
- The intersection of $\sigma(H_{\mathrm{SIBC}})$ and $\sigma(V_g^{-1}H_{\mathrm{PBC}}V_g)$ is strictly smaller than $\sigma(H_{\mathrm{SIBC}})$. This proves that $\sigma(H_{\mathrm{PBC}})\neq\sigma(H_{\mathrm{OBC}})$.
- Furthermore, $\bigcap_{g\in(-\infty,\infty)}\sigma(V_g^{-1}H_{\mathrm{SIBC}}V_g)$ reaches a topological trivial area or curves, otherwise a contradiction arises.

Ex: $1d$ class A with sublattice symmetry

- Sublattice symmetry (non-Hermitian SSH chain)

$$\sigma_z H(k_x) \sigma_z = -H(k_x) \qquad \Rightarrow \qquad H(k_x) = \begin{pmatrix} & h_1(k_x) \\ h_2(k_x) & \end{pmatrix}.$$

- Two $\mathbb{Z}$ topological invariants defined by

$$N_j = \frac{1}{2\pi i} \oint d \log \det h_j(k_x) \in \mathbb{Z} \quad (j = 1, 2).$$

- The classification of point-gap topological phases is $K_{\mathrm{P}} = \mathbb{Z} \oplus \mathbb{Z}$ characterized by (N_1, N_2).
- With the real-line gap condition, $H(k_x)$ can be Hermite, i.e. $h_2(k_x) = h_1(k_x)^\dagger$. The classification of real-line gap topological phases is $K_{\mathrm{L_r}} = \mathbb{Z}$ characterized by $N_1 = -N_2$.
- With the imaginary-line gap condition, $H(k_x)$ can be anti-Hermite, i.e. $h_2(k_x) = -h_1(k_x)^\dagger$. The classification of real-line gap topological phases is $K_{\mathrm{L_i}} = \mathbb{Z}$ characterized by $N_1 = -N_2$.
- Line-gap topology to point-gap topology

$$f_r : K_{\mathrm{L_r}} \to K_{\mathrm{P}}, \quad n \mapsto (n, -n). \quad f_i : K_{\mathrm{L_i}} \to K_{\mathrm{P}}, \quad n \mapsto (n, -n).$$

- Note that the union of images Im $f_r \cup$ Im $f_i = \mathbb{Z}[1, -1] \subset K_{\mathrm{P}}$ does not show the skin effect, since the total phase winding $N_1 + N_2$ is zero.

- $1d$ class A
  - $K_{\mathrm{L}} \to K_{\mathrm{P}}: \quad 0 \to \mathbb{Z}$.
  - Skin effect.

- $1d$ class D
  - $K_{\mathrm{L}} \to K_{\mathrm{P}}: \quad \mathbb{Z}_2 \to \mathbb{Z}_2, \quad 1 \mapsto 1 \quad \Rightarrow \quad K_{\mathrm{P}}/\mathrm{Im}\, f = 0$.
  - No new phenomena unique to non-Hermitian systems.

- $1d$ class AII†
  - Symmetry: $\quad \sigma_y H(k_x)^T \sigma_y = H(-k_x)$.
  - $K_{\mathrm{L}} \to K_{\mathrm{P}}: \quad 0 \to \mathbb{Z}_2$.
  - $\mathbb{Z}_2$ skin effect protected by class AII† TRS! Okuma-Kawabata-KS-Sato

$1d$ class AII†

- $H(k) - E$ is also invariant under TRS†, $\sigma_y[H(k) - E]^T \sigma_y = H(-k) - E$.
- The non-Hermite $\mathbb{Z}_2$ number

$$(-1)^{\nu(E)} = \mathrm{sgn}\left\{ \frac{\mathrm{Pf}[(H(\pi) - E)\sigma_y]}{\mathrm{Pf}[(H(0) - E)\sigma_y]} \times \exp\left[-\frac{1}{2} \int_0^\pi d \log \det[(H(k) - E)\sigma_y] \right] \right\}$$

- Toeplitz index theorem:

$$\#[\text{right zero mode of } H - E] = \nu(E) \mod 2.$$

- Kramers pair: localized right-state .
- We have the dense spectrum protected by the TRS.