AHSS and Band theory

2019/10/24 @RIKEN

Plan

Part 1

- K^0, K^1, K^2 in band theory
- bdy map
- Exactness
- Mayer-Vietoris
- Bott periodicity

Part 2

- Atiyah-Hirzebruch Spectral Sequence (AHSS)
- differentials

$K^0(X)$

in this note
"Hamiltonian"
implies Hermitee

Karoubi's triple

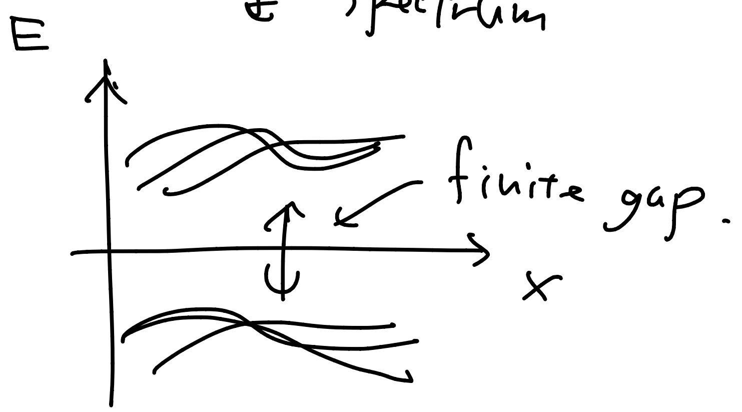
(E, H_0, H_1) "gapped Hamiltonian on E".

complex vector bundle over X.

ex $E = X \times \mathbb{C}^N$, triv bundle.

$H_j(k) = [\ * \ ]$: $N \times N$ Hermite matrix,
$(j=0,1)$

spectrum



$(E, H_0, H_1) \sim (E', H_0', H_1')$

$\overset{\text{def}}{\Longleftrightarrow}$ ${}^\exists (F, H'', H''), (G, H''', H''')$ s.t.

$(E, H_0, H_1) \oplus (F, H'', H'') \cong (E', H_0', H') \oplus (G, H''', H''')$.

$\{E, H_0, H_1\} / \sim \; =: K^0(X)$.

$\longrightarrow$ Abel group.

$0 = [E, H, H]$.

$-[E, H_0, H_1] = [E, H_1, H_0]$.

ex $K^0(pt) = \mathbb{Z}$.

$1 = (\mathbb{C}, H_0 = -1\ H_1 = 1)$

☆ WLOG, E can be trivial bundle.

$\because$ $E \oplus E^{\perp} \cong X \times \mathbb{C}^N$.

$K^0(X)$

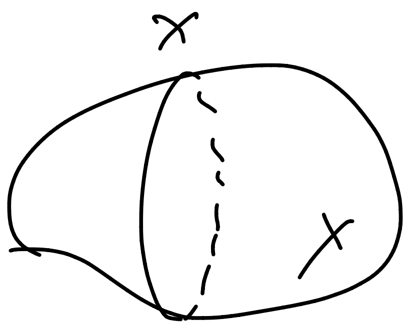
$\sim$ "gapped Hamiltonian over X".

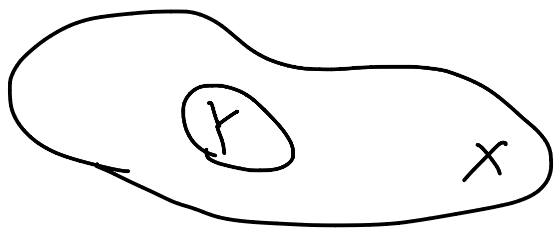
Axiom of generalized cohomology...

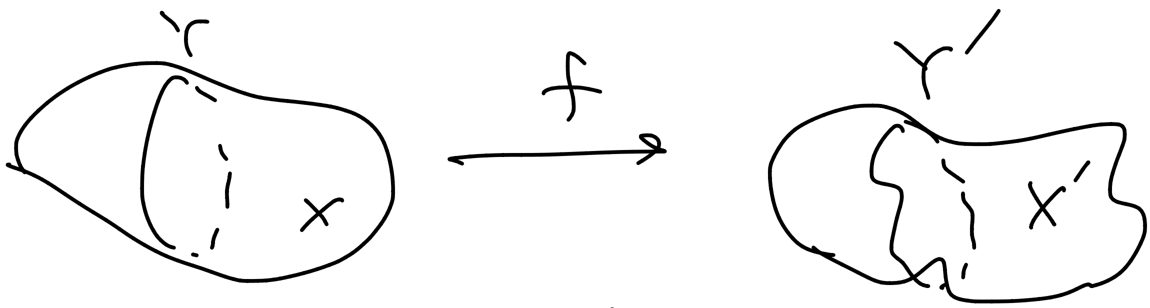
$n \in \mathbb{Z}$, $(X, Y) \longrightarrow K^n(X, Y)$,

contravariant functor.

(X, Y) : $Y \subset X$







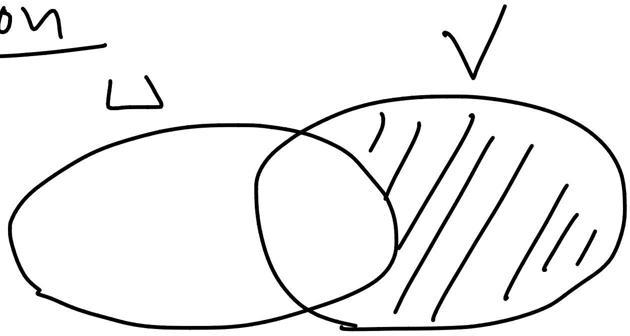
$K^n(X, Y) \xleftarrow{f^*} K^n(X', Y')$

(i) homotopy

$$(X, Y) \underset{f'}{\overset{f}{\rightrightarrows}} (X', Y') \quad f \sim f'$$

$$\Rightarrow \quad f^* = f'^*$$

(ii) excision



$$K^n(U \cup V, U) \cong K^n(V, U \cap V).$$

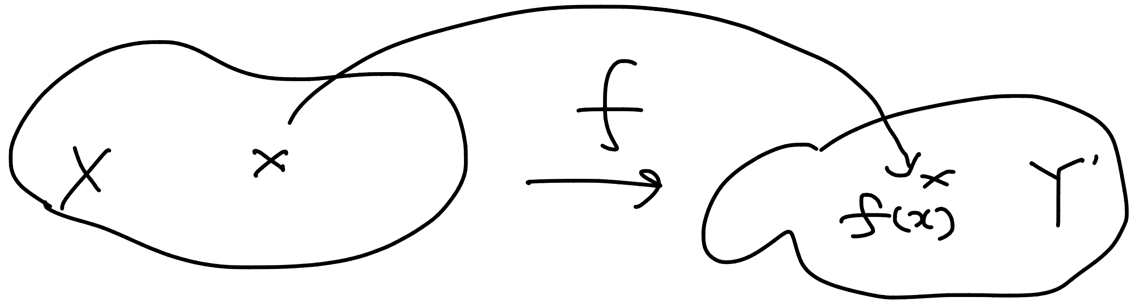
(iii) additivity

$$K^n\left(\coprod_\lambda X_\lambda\right) = \bigoplus_\lambda K^n(X_\lambda)$$

(iv) exactness

$$\rightarrow K^n(X, Y) \rightarrow K^n(X) \rightarrow K^n(Y) \rightarrow K^{n+1}(X, Y) \rightarrow \cdots$$

@ contravariant?



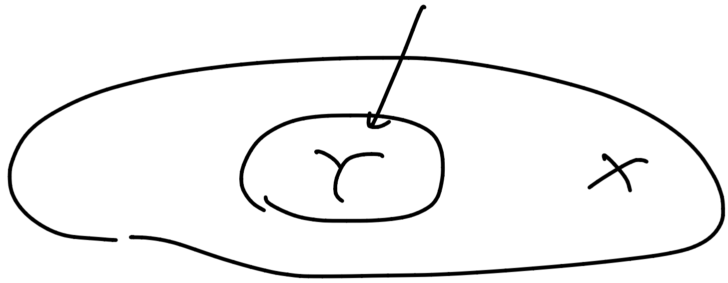
$$K^0(X) \longleftarrow K^0(Y)$$

$$H_X(x) := H_Y(f(x)) \longleftarrow H_Y(y \in Y) \quad //$$

@ $K^0(X,Y) = ?$

⇝ gapped Hamiltonian over X which is "<u>trivial</u>" over Y.

$H_0(x) = H_1(x),\ x \in Y.$



• exactness.

$$\to K^0(X,Y) \xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(Y) \xrightarrow{d} K^1(X,Y) \to$$

$$\left\{\begin{array}{l} H(x\in X), \\ H(x\in Y) = \text{triv.} \end{array}\right. \mapsto H(x\in X) \mapsto H(x\in Y) \mapsto ?$$

$K^1(X)$: gapless Hamiltonian over X,

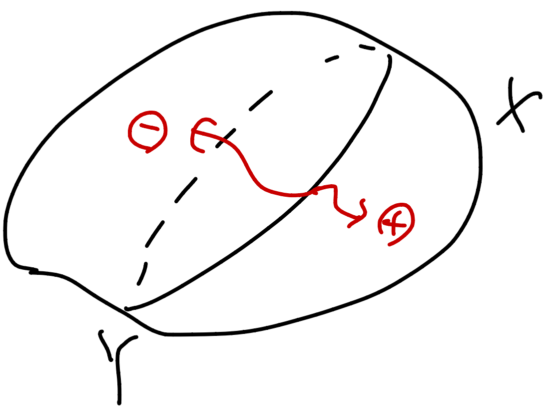
or.

"spectral flow"

$K^1(X,Y)$; gapless Hamiltonian over X which is gapped over Y,

$$d : K^0(Y) \to K^1(X,Y)$$

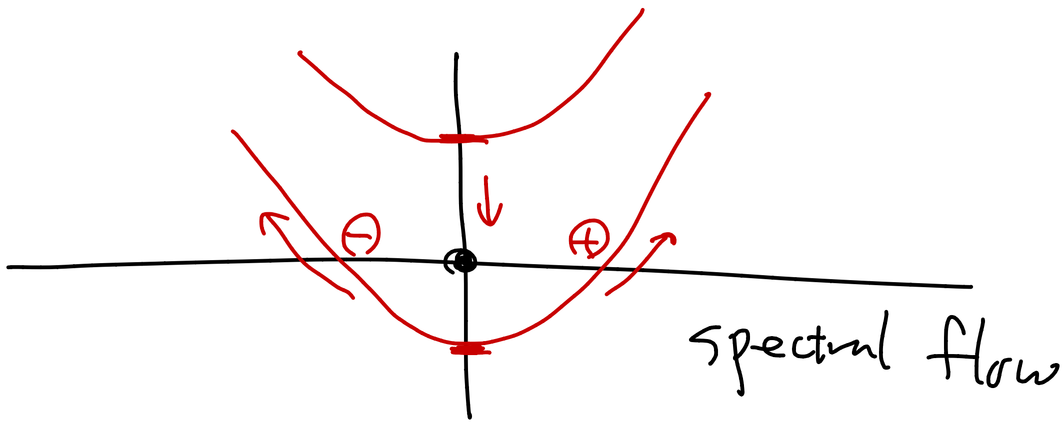
def $\iff$ $\boxed{\text{"topological phase transition" on } Y \text{ and "creating gapless pts." over } X.}$



show Exactness

-- exercise.

ex



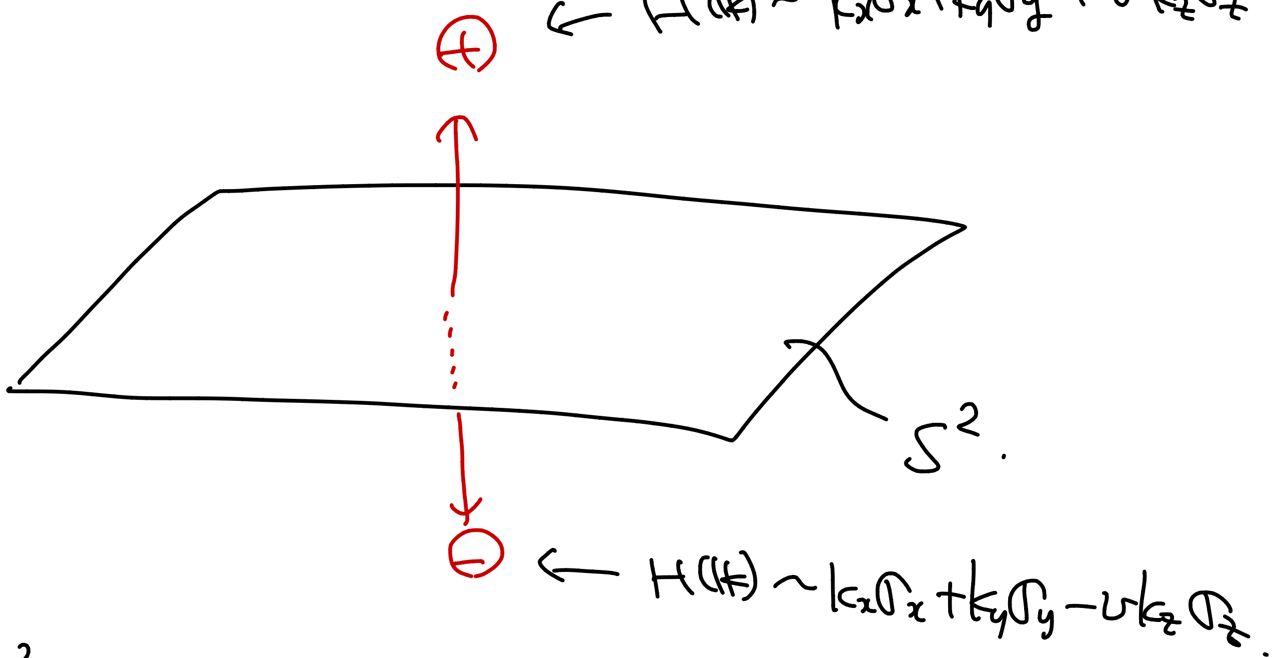
$Y = pt.$

$$K^0(pt) = \mathbb{Z} \ni 0 = [\mathbb{C},\ H_0 = 1,\ H_1 = 1$$

$$\downarrow$$

$$1 = [\mathbb{C},\ H_0 = -1,\ H_1 = 1] .$$

ex



$$K^0(S^2) = \mathbb{Z} \times \mathbb{Z} \ni [\mathbb{C}, 1, -1]$$

"Chern insulator"

$k_x^2 + k_y^2 + k_z^2$.

$$\left[S^2 \times \mathbb{C}^2,\ \underset{k_x, k_y, k_z}{H_0(k)} = k_x\sigma_x + k_y\sigma_y + (m + \epsilon k^2)\sigma_z,\ H_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\right]$$

$m > 0$, $\epsilon > 0$.

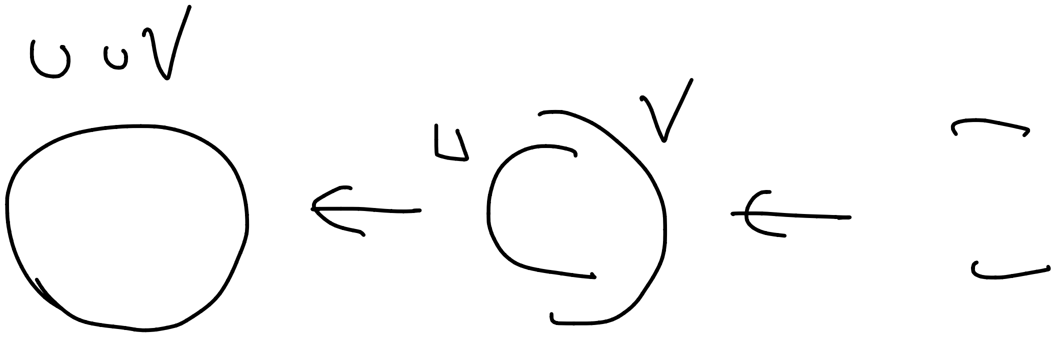
$\downarrow$ $ch = 0$, Chern #

$$H_0(k) = k_x\sigma_x + k_y\sigma_y + (m + \epsilon k^2)\sigma_z.$$

$m < 0$ $\epsilon > 0$.

$ch = 1$.

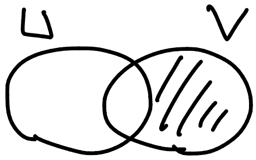
Mayer-Vietris sequence.



$\Downarrow$ long exact sequence

$$\to K^n(U \cup V) \to K^n(U) \oplus K^n(V) \to K^n(U \cap V) \to K^{n+1}(U \cup V) \to \cdots$$

(Proof)



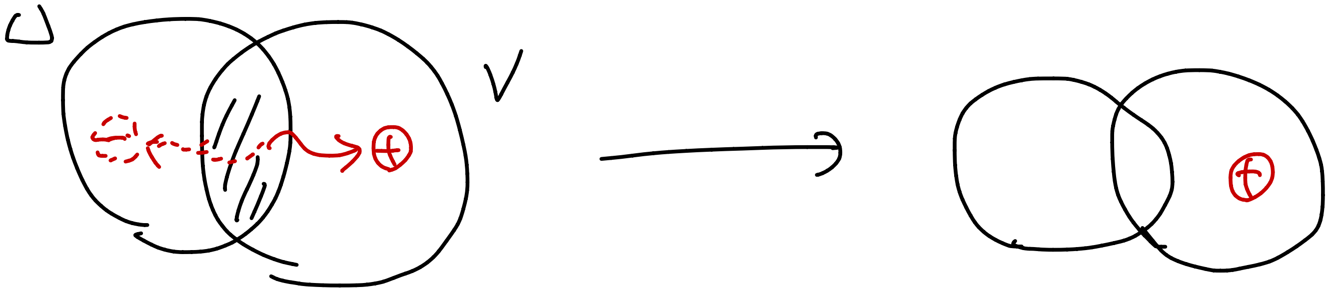
$V, U\cap V$

$$\begin{tikzcd}
\to K^n(U\cup V, U) \arrow[r, "\tilde{i}^*_{(U\cup V, U)}"] \arrow[d, "\cong"] & K^n(U\cup V) \arrow[r, "\tilde{i}^*_U"] \arrow[d, "i^*_V"] & K^n(U) \arrow[r] \arrow[d, "j^*_U"] & K^{n+1}(U\cup V, U) \to \arrow[d, "\cong"] \\
\to K^n(V, U\cap V) \arrow[r] & K^n(V) \arrow[r, "j^*_V"] & K^n(U\cap V) \arrow[r, "d"] & K^{n+1}(V, U\cap V) \to
\end{tikzcd}$$

$\Downarrow$ (Barratt-Whitehead)

$$\to K^n(U\cup V) \xrightarrow{i^*_U,\, i^*_V} K^n(U)\oplus K^n(V) \xrightarrow{j^*_U - j^*_V} K^n(U\cap V) \xrightarrow{\delta} K^{n+1}(U\cup V) \to$$

$$\delta : \ K^0(U\cap V) \xrightarrow{d} K^1(V, U\cap V) \underset{\cong}{\to} K^1(U\cup V, U) \xrightarrow{\tilde{i}^*_{(U\cup V, U)}} K^1(U\cup V)$$



δ : top. phase transition on $U\cap V$,
focusing only on gapless pt on V.

ex

$$(n, m) \longleftrightarrow (n-m, n-m)$$

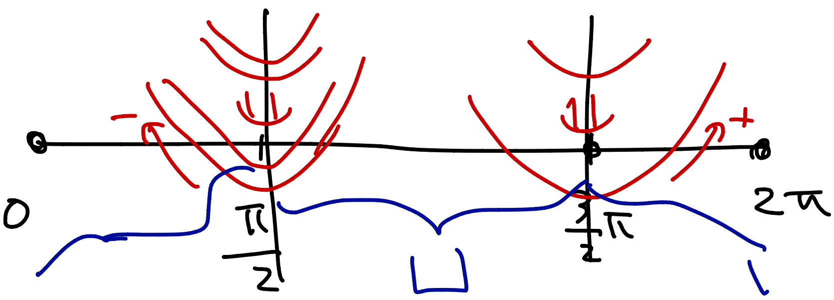
$$\to K^0(S^1) \to \overset{\mathbb{Z}}{\overset{\shortparallel}{K^0(U)}} \oplus \overset{\mathbb{Z}}{\overset{\shortparallel}{K^0(V)}} \xrightarrow{f} \overset{\mathbb{Z}\oplus\mathbb{Z}}{\overset{\shortparallel}{K^0(U\cap V)}}$$

$$\xrightarrow{\delta^0} \overset{=\mathbb{Z}}{K^1(S^1)} \to \underset{0}{\underset{\shortparallel}{K^1(U)}} \oplus \underset{0}{\underset{\shortparallel}{K^1(U)}} \to \underset{0}{\underset{\shortparallel}{K^1(U\cap V)}}.$$

$$\to K^2(S^1) \to \cdots$$

$$\delta^0 : \underset{\mathbb{Z}\oplus\mathbb{Z}}{\underset{\shortparallel}{K^0(U\cap V)}} \longrightarrow K^1(S^1).$$

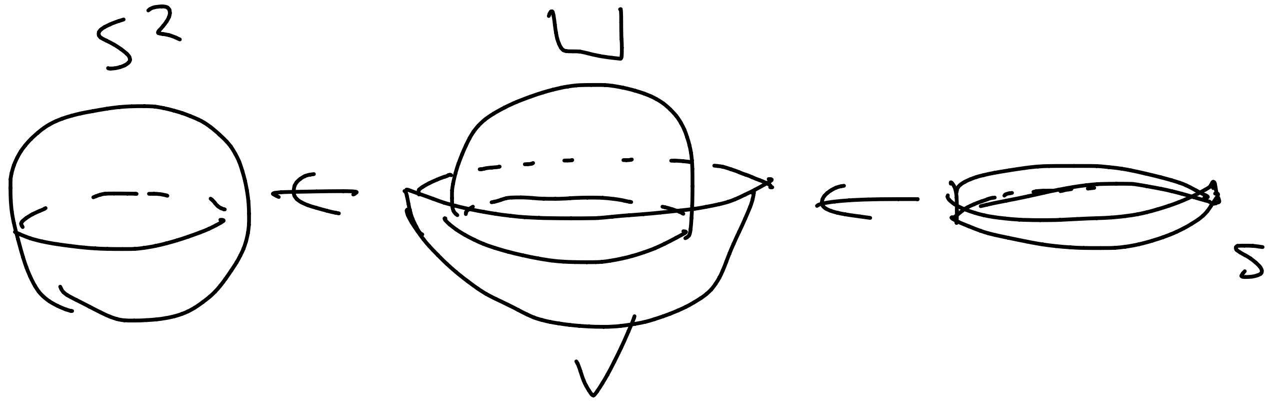
$$\mapsto -n+m,$$



※ Bott periodicity $K^{n+2} \cong K^n$ (later)

$$\to K^0(S^1) \cong \operatorname{Ker} f = \mathbb{Z}.$$

ex



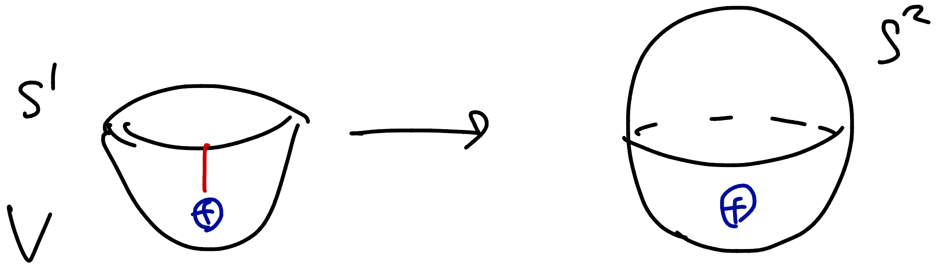
$$\to K^0(S^2) \to \overset{\mathbb{Z}}{\overset{\parallel}{K^0(U)}} \oplus \overset{\mathbb{Z}}{\overset{\parallel}{K^0(V)}} \to \overset{\mathbb{Z}}{\overset{\parallel}{K^0(S^1)}}$$

$$\to K^1(S^2) \to \overset{0}{\overset{\parallel}{K^1(U)}} \oplus \overset{0}{\overset{\parallel}{K^1(V)}} \to \overset{\mathbb{Z}}{\overset{\parallel}{K^1(S^1)}}$$

$$\xrightarrow{\delta} K^2(S^2) \to K^2(U) \oplus K^2(V) \to K^2(S^1) \to \cdots$$

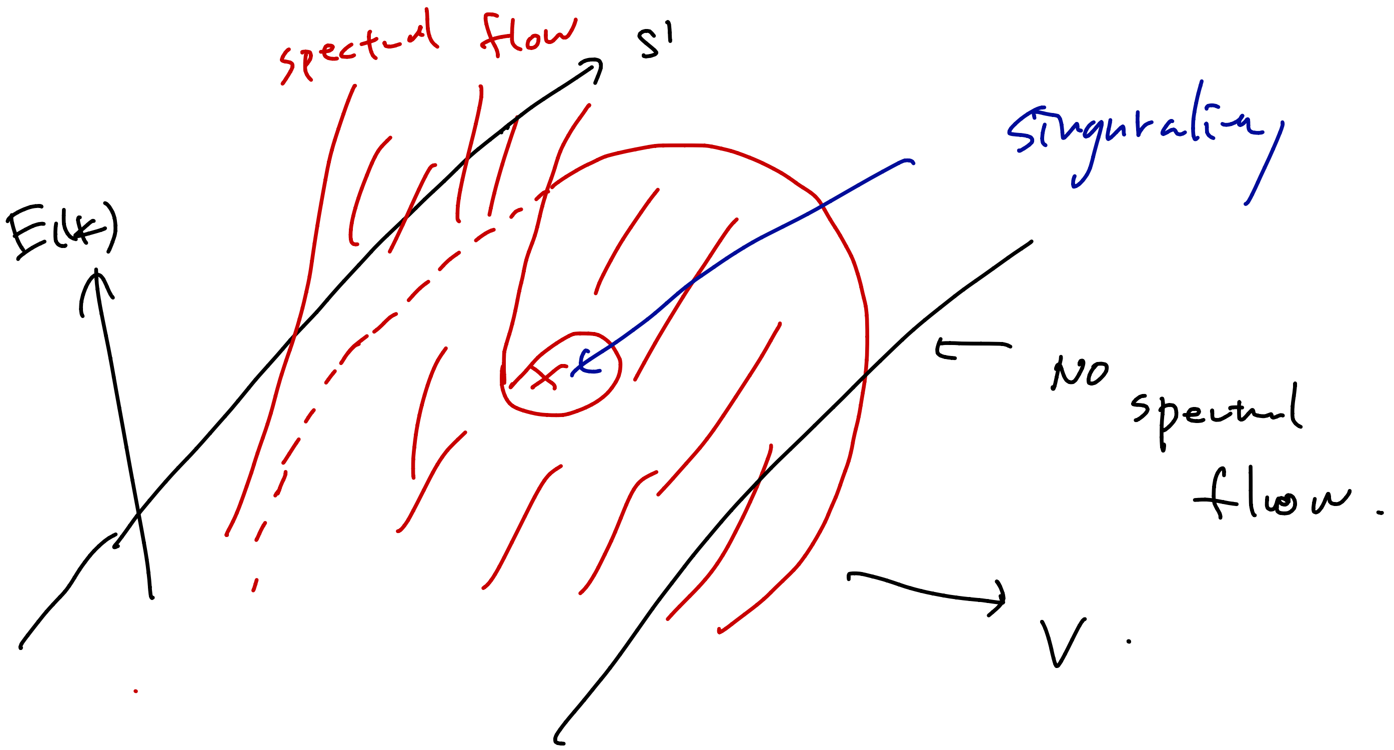
⊚ $\underset{\text{gapless Hamiltonian}}{K^1(S^1)} \xrightarrow{\delta} K^2(S^2)$

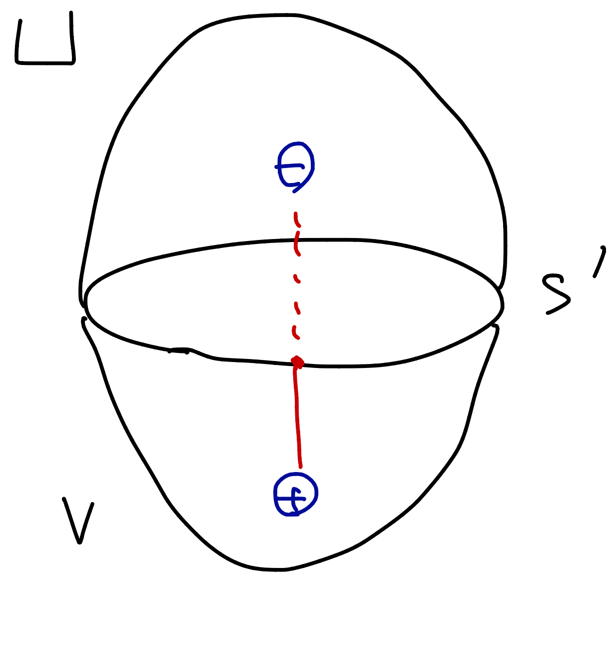
?



$K^2(X) \ni$ singular Hamiltonian over X !

$K^2(X,Y) =$ singular Hamiltonian over X which has no singularity over Y.


spectral flow
S^1
singularity
$E(k)$
No spectral flow.
V.


U
S^1
V
$\ominus$
$\oplus$

Grading and chiral sym

$K^1(X)$: gapless Hamiltonian over X.

= unitary matrix over X !

gapless

$\because$ $H(k_x) = \mathrm{Im} \log \underset{\text{unitary}}{U(k_x)}$.

ex $K^1(S^1) = \mathbb{Z}$ ← spectral flow.

$H(k_x) \sim k_x = \mathrm{Im} \log \underset{\text{unitary}}{\underline{e^{ik_x}}}$.

= $\boxed{\text{gapped Hamiltonian w/ "chiral sym"}}$

$\Gamma = \sigma_z$, $\{\Gamma, \tilde{H}(k_x)\} = 0$

$\because$ $\tilde{H}(k_x) := \begin{pmatrix} & U(k_x) \\ U(k_x)^\dagger & \end{pmatrix}$, $\Gamma = \sigma_z$.

in general,

$$\boxed{K^n(X) = \text{gapped Hamiltonian w/ } n \text{ chiral symmetries.}}$$

$$\{H(k), \Gamma_j\} = 0 \quad (j = 1, \cdots, n)$$

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}.$$

ex $K^2(x)$

$K^2(X)$ = gapless Hamiltonian w/ chiral sym.

$$H(k), \ \{\Gamma, H(k)\} = 0.$$

= unitary matrix w/ sym. $\Gamma U(k) \Gamma^{-1} = U(k)^\dagger$.

$$H(k) = \operatorname{Im} \log U(k).$$

= gapped Hamiltonian w/ two chiral syms.

$$\tilde{H}(k) = \begin{pmatrix} & U(k) \\ U(k)^\dagger & \end{pmatrix}_\sigma,$$

$$\tilde{\Gamma}_1 = \sigma_y \Gamma, \quad \tilde{\Gamma}_2 = \sigma_z. \quad \{\tilde{\Gamma}_1, \tilde{\Gamma}_2\} = 0.$$

Bott periodicity

$$K^{n+2} \cong K^{n}.$$

$\because$ gapped Hamiltonian w/ 2 chiral syms.

$\Gamma_1 = \sigma_x, \ \Gamma_2 = \sigma_y, \ \{H, \Gamma_j\} = 0, \ (j = 1, 2)$

$$\Rightarrow \quad H = \tilde{H} \otimes \sigma_z$$

$\tilde{H}$: gapped Hamiltonian w/o chiral sym.

//

Take home message

2
|

- $K^{0}(X,Y)$: singular gapless / gapped Hamiltonian over X

  which is regular gapped / trivial over Y

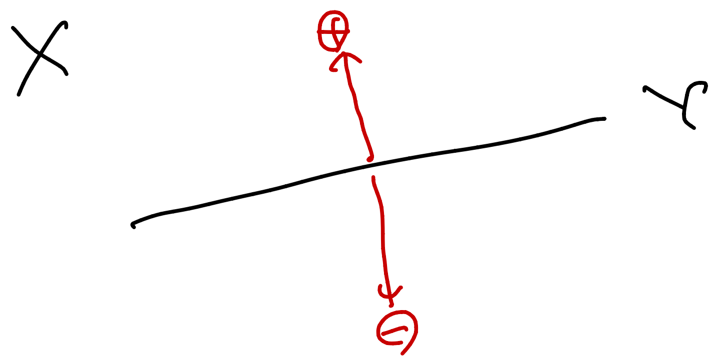
  $=$ K^{n} : gapped Hamiltonian w/ n chiral syms.

- bdy map

$$K^{n}(Y) \longrightarrow K^{n+1}(X,Y)$$

$=$ "top. phase transition" over Y

and "creating gapless pts" over X.



Part 1

Part II (arXiv: 1802.06694)

Spectral sequence

$h^*(X,Y)$: a generalized cohomology theory.

$E_1^{p,q}$: something computable

$$E_1^{p,q} \Rightarrow E_2^{p,q} \Rightarrow E_3^{p,q} \Rightarrow \cdots \Rightarrow E_\infty^{p,q}$$

d_1 (differential), cohomology, d_2

$$E_\infty^{p,q} \longrightarrow h^{p+q}(X,Y).$$

approximate.

K-theory for the band theory

$${}^{\phi}K_G^{(\tau,c)-n}(X)$$

- G. finite group $\curvearrowright X$ $(X = T^2, T^3)$
- $\phi : G \to \mathbb{Z}_2$: $\begin{cases} \text{unitary} \\ \text{anti unitary} \end{cases}$.
- $c : G \to \mathbb{Z}_2$: $\hat{g} H \hat{g}^{-1} = \underset{c}{\pm} H$.
- τ : "factor system"

  $$\hat{g}\hat{h} = e^{i\tau_{g,h}} \widehat{gh}. \quad g, h \in G.$$

  ($\hat{g}$, $\tau_{g,h}$ depends on $k \in X$).
- $n \in \mathbb{Z}$, grading.

  (Bott periodicity $K^{-n} \cong K^{-n+8}$).
- $n \mapsto n+1$:

  add a chiral symmetry γ, $\gamma \hat{g} \gamma^{-1} = c_g \hat{g}$.

How to compute ${}^{\phi}K_G^{(\tau,c)-n}(X)$? $\longrightarrow$ AHSS.

comment.

$G_T = \mathbb{Z}_2 = \{1, T\}$, $\phi_T = -1$, antiunitary

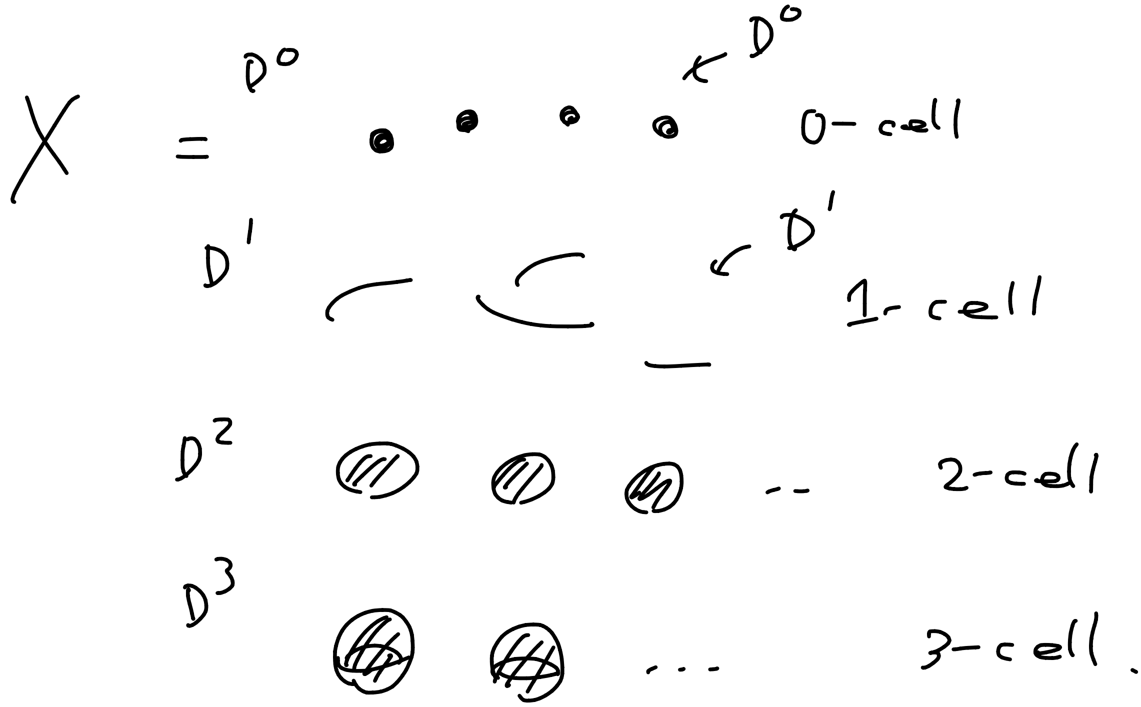
$c_T = 1$, sym.

$$THT^{-1} = H.$$

	T	C	Altland-Zirnbauer class (AZ)
K^{-0}	$T^2=1$		AI
K^{-1}	$T^2=1$	$C^2=1$	BDI
K^{-2}		$C^2=1$	D
K^{-3}	$T^2=-1$	$C^2=1$	DIII
K^{-4}	$T^2=-1$		AII
K^{-5}	$T^2=-1$	$C^2=-1$	CII
K^{-6}		$C^2=-1$	C
K^{-7}	$T^2=1$	$C^2=-1$	CI.
	↑ TRS	↑ PHS.	

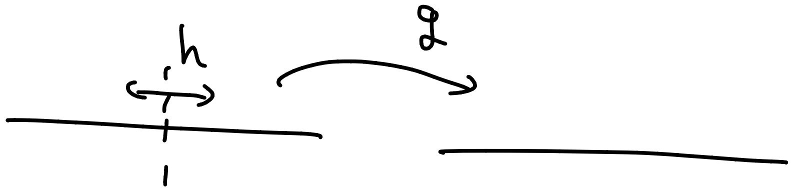
AHSS for an "high-symetric cell decomposition"

- Cell decomposition :



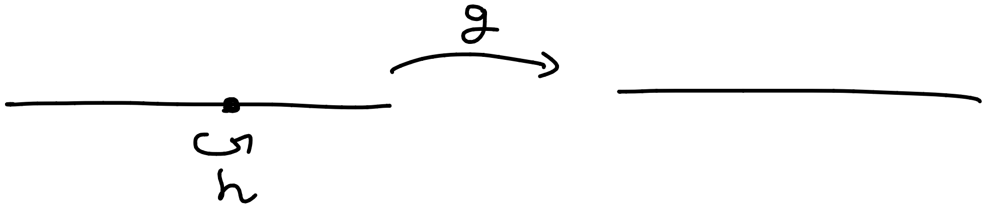
- G-sym cell decomposition

$\Leftrightarrow$ G acts on p-cells freely or symmetric



- "high-symetric cell decomposition"

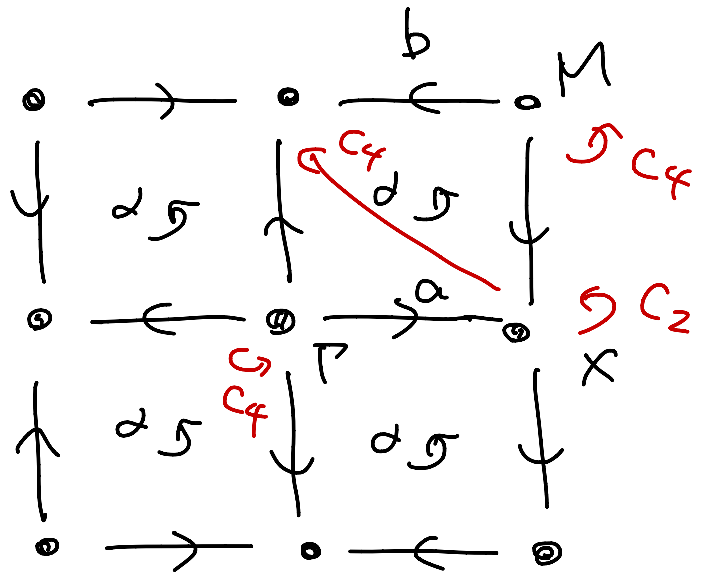
$\Leftrightarrow$ G acts on p-cells freely or trivially.

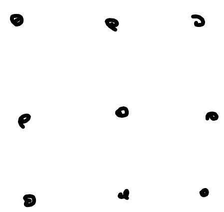


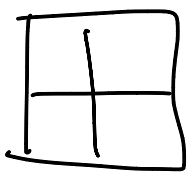
• p-skeleton

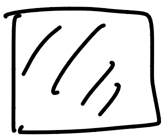
$$X_p = X_{p-1} \cup \{p\text{-cells}\}.$$

ex $\quad C_4 \curvearrowright T^2. \quad (k_x, k_y) \mapsto (-k_y, k_x).$



$X_0 =$ 

$X_1 =$ 

$X_2 =$  $= T^2.$

take-home message. differentials of AHSS for a high-symmetric cell decomposition is computable.

(on going)

AHSS

- $$E_1^{p,-n} = K^{p-n}(X_p, X_{p-1})$$

physical meaning

$$= \boxed{\bigoplus_{j \in p\text{-cells}} K^{p-n}(D^p_j, \partial D^p_j)}$$

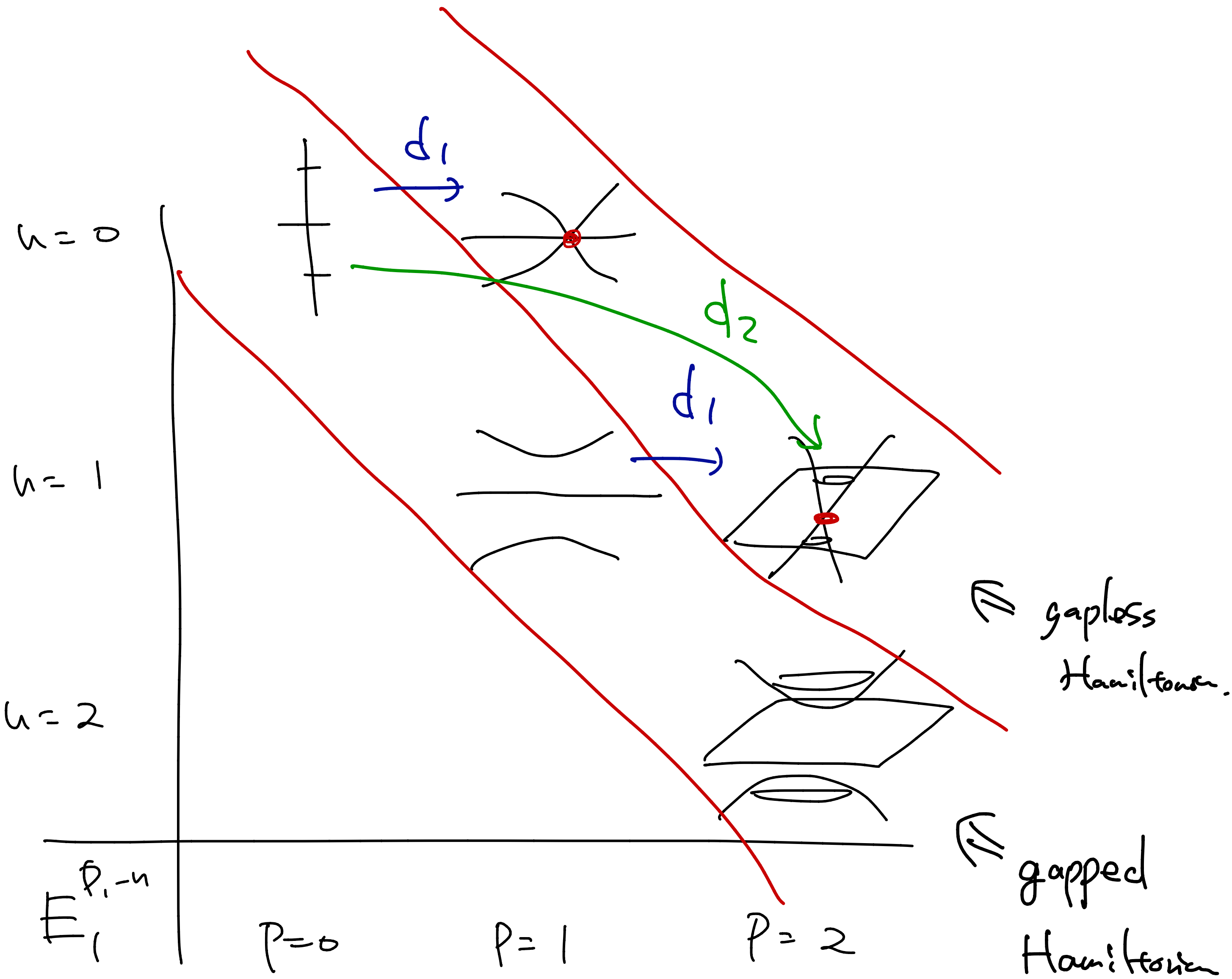
S^p

$$= \bigoplus_{j \in p\text{-cells}} \tilde{K}^{p-n}(D^p_j / \partial D^p_j)$$

$$= \bigoplus_{j \in p\text{-cells}} K^{-n}(pt). \quad (\because \text{suspension iso})$$

$$K^{-n}(pt) = \bigoplus_{\text{irreps}} \mathbb{Z} \text{ or } \mathbb{Z}_2$$

$\Rightarrow$ easy to compute.



n = 0
n = 1
n = 2
$E_1^{p,-n}$
p = 0
p = 1
p = 2
d_1
d_2
d_1
gapless Hamiltonian.
gapped Hamiltonian

• First differential d_1

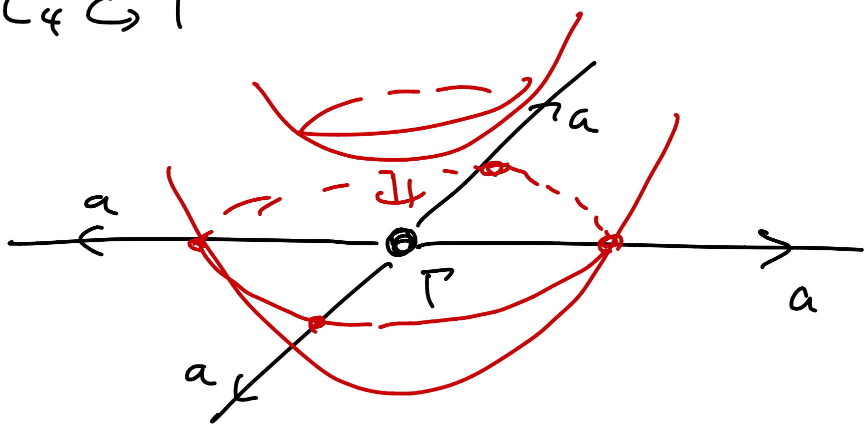
$$d_1^{p,-n} : E_1^{p,-n} \to E_1^{p+1,-n}$$

$$E_1^{p,-n} = K^{p-n}(D^p_{\bar{j}}, \partial D^p_{\bar{j}}) \qquad E_1^{p+1,-n} = K^{p+1-n}(D^{p+1}_{\bar{i}}, \partial D^{p+1}_{\bar{i}}).$$

= "top phase transition" @ p-cell $D^p_{\bar{j}}$

and "creating gapless pts" on $D^{p+1}_{\bar{i}}$.

ex $C_4 \curvearrowright T^2$.



model Hamiltonian.

$$H(k) = (k^2 - \mu)\,\mathbb{1}_d. \qquad d : \text{irrep @ } A.$$

$$C_4 = \begin{cases} 1 \\ -1 \\ +i \\ -i \end{cases}.$$

E_2-page

$d_1 \circ d_1 = 0$

$\Rightarrow$ $\ker d_1 / \mathrm{Im}\, d_1$ is well-defined.

$$E_2^{p,-n} := \ker d_1^{p,-n} / \mathrm{Im}\, d_1^{p-1,-n} .$$

= gapped Hamiltonian on p-cells
which has no gapless pts on (p+1)-cells,
(ker)

and is not trivialized by (p−1)-cells.

Second differential d_2

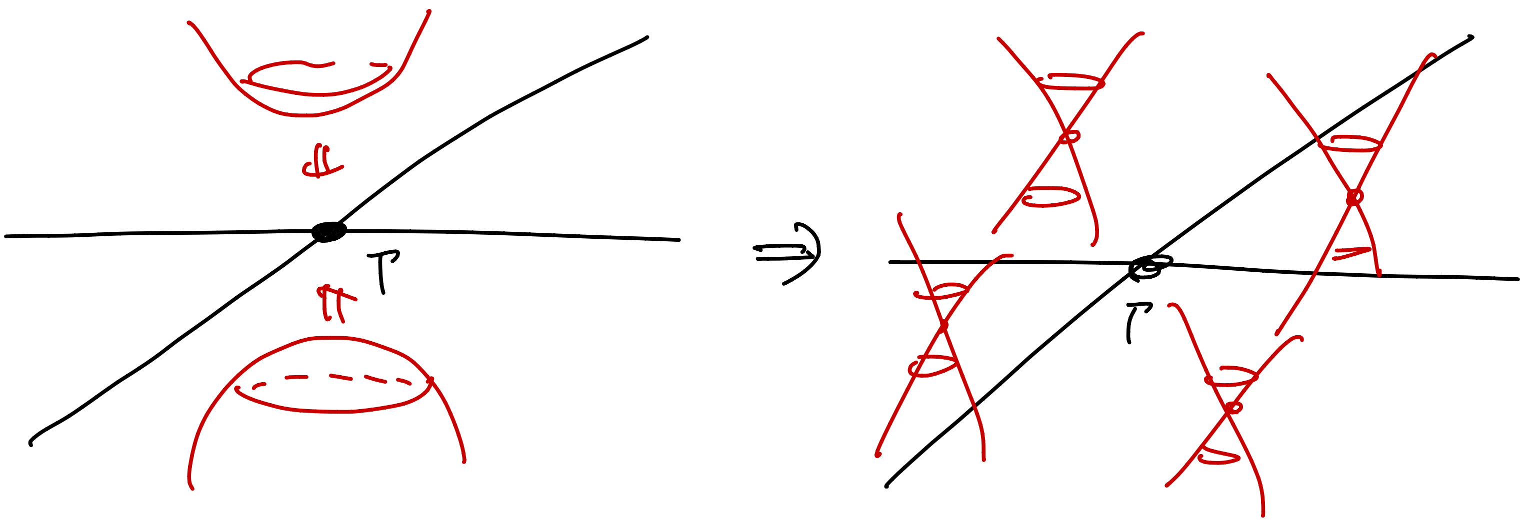
$$d_2^{p,-n} : E_2^{p,-n} \longrightarrow E_2^{p+2,-n+1}$$

= "top. phase transition" @ p-cells

and "creating gapless pts" @ (p+2)-cells.

"band inversion"

ex. $C_4 + \alpha$



model Hamiltonian

$$H(k_x,k_y) = (k_x^2-k_y^2)\gamma_1 + (k^2-\mu)\gamma_2 .$$

in the same way, we define

r-th differential d_r and E_{r+1}-page

$$E_{r+1}^{p,q} = \operatorname{Ker} d_r^{p,q} \,/\, \operatorname{Im} d_r^{p-r,\, q+r-1}.$$

$$E_1 \Rightarrow E_2 \Rightarrow E_3 \Rightarrow \cdots \Rightarrow E_\infty .$$

◎ $E_\infty^{p,-n}$ - page approximates $K^{p-n}(X)$.

$E_\infty^{p,-n}$: gapped Hamiltonians over p-cells which have no gapless pts on any $(p+r)$-cells, and can not trivialized by any $(p-r)$-cells.

- $F^p K^{-n} := \mathrm{Ker}\left[K^{-n}(X) \xrightarrow{i^*} K^{-n}(X_{p-1}) \right]$.

$$X_{p-1} \hookrightarrow X.$$

= gapped Hamiltonian over X

which is trivial over X_{p-1}.

$$\subset F^{p+1}K^{-n} \subset F^p K^{-n} \subset \cdots \subset F^0 K^{-n} = K^{-n}(X)$$

$$\Uparrow$$

$$E_\infty^{p,-p-n} = F^p K^{-n} / F^{p+1} K^{-n}.$$

$$\Uparrow$$

gapped Hamiltonian over X

which is trivial over X_{p-1}

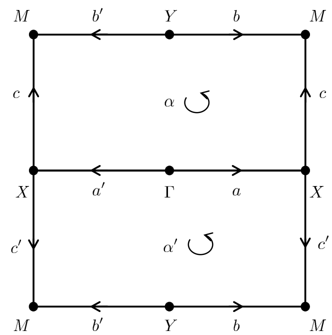
and nontrivial over X_p.

$$= E_\infty^{p,-p-n}.$$

ex　(時間余れば).

B. A warm up: real AZ class in two space dimensions

As a warm up, let us compute the AHSS for $2d$ system with real AZ symmetry classes. As the symmetry class for $n = 0$, we consider the $2d$ spinless systems with the TRS T. The symmetry group is $G = \mathbb{Z}_2 = \{1, T\}$ which acts on the $2d$ BZ torus by $T : (k_x, k_y) \mapsto (-k_x, -k_y)$. The factor system is $T^2 = 1$. We use the following $\mathbb{Z}_2$-filtration of the BZ torus:



(4.16)

Here, $a' = T(a), b' = T(b), \alpha' = T(\alpha)$ represent equivalent p-cells. From (4.3), the E_1-page is given by the space of representations at p-cells with symmetry class determined by $n \in \mathbb{Z}$.

$$\begin{array}{cc|ccc}
\mathrm{AI} & n=0 & \mathbb{Z}^4 & \mathbb{Z}^3 & \mathbb{Z} \\
\mathrm{BDI} & n=1 & \mathbb{Z}_2^4 & 0 & 0 \\
\mathrm{D} & n=2 & \mathbb{Z}_2^4 & \mathbb{Z}^3 & \mathbb{Z} \\
\mathrm{DIII} & n=3 & 0 & 0 & 0 \\
\mathrm{AII} & n=4 & \mathbb{Z}^4 & \mathbb{Z}^3 & \mathbb{Z} \\
\mathrm{CII} & n=5 & 0 & 0 & 0 \\
\mathrm{C} & n=6 & 0 & \mathbb{Z}^3 & \mathbb{Z} \\
\mathrm{CI} & n=7 & 0 & 0 & 0 \\
\hline
 & E_1^{p,-n} & p=0 & p=1 & p=2
\end{array} \tag{4.17}$$

The first differential $d_1^{p,-n} : E_1^{p,-n} \to E_1^{p+1,-n}$ is straightforwardly given by the compatibility relation as in

$$d_1^{0,0} = \begin{array}{cccc|c}
\Gamma & X & Y & M & \\
\hline
1 & -1 & 0 & 0 & a \\
0 & 0 & 1 & -1 & b \\
0 & 1 & 0 & -1 & c
\end{array}, \qquad
d_1^{0,-4} = \begin{array}{cccc|c}
\Gamma & X & Y & M & \\
\hline
2 & -2 & 0 & 0 & a \\
0 & 0 & 2 & -2 & b \\
0 & 2 & 0 & -2 & c
\end{array}, \qquad
d_1^{1,-2} = d_1^{1,-6} = \begin{array}{ccc|c}
a & b & c & \\
\hline
2 & -2 & 0 & \alpha
\end{array}, \tag{4.18}$$

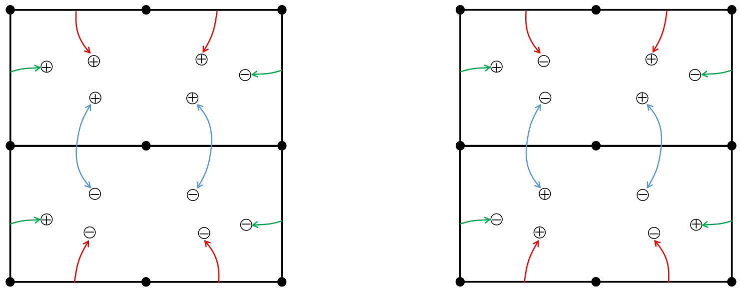
and $d_1^{p,-n} = 0$ for others. We should be careful about the PHS. For $n = 2$ and 6, the 1-cell a (b) changes to a' (b') with the particle-hole transformation C. Then, an occupied state $|\phi\rangle$ at a is sent to an empty state $C|\phi\rangle$ at a', which results in a nontrivial first differential.

Before moving to the second differential d_2, it is worth understanding the first differentials in terms of gapless Dirac points. According to the meaning (II) of the E_1-page in Sec. IV A, $E_1^{p,-n}$ can be views as the space of topological gapless points inside p-cells with the symmetry class $(n - p + 1)$:

AI	$n=0$	$\mathbb{Z}^4$	$\mathbb{Z}^3$	$\mathbb{Z}$	← CI gapless points
BDI	$n=1$	$\mathbb{Z}_2^4$	0	0	← AI gapless points
D	$n=2$	$\mathbb{Z}_2^4$	$\mathbb{Z}^3$	$\mathbb{Z}$	← BDI gapless points
DIII	$n=3$	0	0	0	← D gapless points
AII	$n=4$	$\mathbb{Z}^4$	$\mathbb{Z}^3$	$\mathbb{Z}$	← DIII gapless points
CII	$n=5$	0	0	0	← AII gapless points
C	$n=6$	0	$\mathbb{Z}^3$	$\mathbb{Z}$	← CII gapless points
CI	$n=7$	0	0	0	← C gapless points
	$E_1^{p,-n}$	$p=0$	$p=1$	$p=2$	

Here, $E_1^{2,\mathrm{even}} = \mathbb{Z}$ is understood from that the chiral symmetry stabilizes a Dirac point in a 2-cell. The first differentials $d_1^{1,\mathrm{even}} : E_1^{1,\mathrm{even}} \to E_1^{2,\mathrm{even}}$ represent how gapless Dirac points in the 2-cell are absorbed by the pair creation of the

Dirac points from 1-cells. For class BDI/CII, the time-reversal symmetric pair of Dirac points has the opposite charge because of the algebra $T\Gamma = \Gamma T$ between the TRS T and the chiral symmetry Γ, whereas for class DIII/CI, the charge is the same because of the algebra $T\Gamma = -\Gamma T$. This accounts for $d_1^{1,(-2)/(-6)} = (2,-2,0)$ and $d_1^{1,0/(-4)} = (0,0,0)$ as shown below:



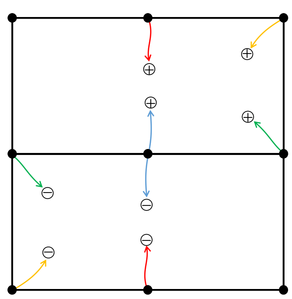
BDI/CII DIII/CI

Taking the cohomology of d_1, we have the E_2-page

$$
\begin{array}{cc|ccc}
\mathrm{AI} & n=0 & \mathbb{Z} & 0 & \mathbb{Z} \\
\mathrm{BDI} & n=1 & \mathbb{Z}_2^4 & 0 & 0 \\
\mathrm{D} & n=2 & \mathbb{Z}_2^4 & \mathbb{Z}^2 & \mathbb{Z}_2 \\
\mathrm{DIII} & n=3 & 0 & 0 & 0 \\
\mathrm{AII} & n=4 & \mathbb{Z} & \mathbb{Z}_2^3 & \mathbb{Z} \\
\mathrm{CII} & n=5 & 0 & 0 & 0 \\
\mathrm{C} & n=6 & 0 & \mathbb{Z}^2 & \mathbb{Z}_2 \\
\mathrm{CI} & n=7 & 0 & 0 & 0 \\
\hline
 & E_2^{p,-n} & p=0 & p=1 & p=2
\end{array} \tag{4.19}
$$

In the E_2-page, the second differential $d_2^{0,-1} : E_2^{0,-1} \to E_2^{2,-2}$ can be nontrivial. Actually, we find that $d_2^{0,-1}$ is surjective. Notice that the even/odd parity of the class BDI $1d$ winding number $w_{1d}[-a'+a]$ along the loop $-a'+a$ is given by the $\mathbb{Z}_2$ invariants at Γ and X as $(-1)^{w_{1d}[-a'+a]} = (-1)^{\nu(\Gamma)}(-1)^{\nu(X)}$. In the same way, it holds that $(-1)^{w_{1d}[-b'+b]} = (-1)^{\nu(Y)}(-1)^{\nu(M)}$ for the $1d$ winding number along the loop $-b'+b$. Therefore, the product $(-1)^{\nu} := \prod_{\boldsymbol{k}\in\{\Gamma,X,Y,Z\}}(-1)^{\nu(\boldsymbol{k})}$ is the $\mathbb{Z}_2$ indicator to detect an odd number of class BDI Dirac points inside the 2-cell α and $(-1)^{\nu}$ is nothing but the second differential $d_2^{0,-1}$.

Alternatively, one can readily find the second differential $d_2^{0,-1}$ in the view of the pair creations of the Dirac points from 0-cells. The class BDI symmetry permits creating a pair of Dirac points from 0-cells which removes the $\mathbb{Z}_2$ remainder of $E_2^{2,-2} = \mathbb{Z}_2$, the odd charges of Dirac points in the 2-cell. See the following figure:



BDI

Explicitly, this pair creation of Dirac points in class BDI can be modeled as

$$
H(\boldsymbol{k}) = (|\boldsymbol{k}-\boldsymbol{k}_0|^2 - \mu)\tau_z + (\boldsymbol{k}-\boldsymbol{k}_0)\cdot\boldsymbol{n}\tau_y, \qquad T = K, \qquad C = \tau_x, \tag{4.20}
$$

around a 0-cell $\boldsymbol{k}_0$. It is clear that when μ passes zero, the band inversion occurs with the change of the $\mathbb{Z}_2$ Pfaffian invariant $(-1)^{\nu(\boldsymbol{k}_0)}$ at the 0-cell $\boldsymbol{k}_0$, and a pair of Dirac points are pumped to the direction perpendicular to $\boldsymbol{n}$.

This can contrast well with the case of class CII, where the pair creation of Dirac points from 0-cells should be doubly degenerate due to the TRS with Kramers. As a result, there is no class CII topological invariant at 0-cells (this is the meaning of that $E_1^{0,-5}=0$), which means that the 0-cells can not be a new source of Dirac points, i.e. Dirac points arising from the 0-cells are recast as ones from 1-cells.

Taking the cohomology of d_2, we arrive at the limiting page $E_\infty = E_3$,

$$\begin{array}{cc|ccc}
\mathrm{AI} & n=0 & \mathbb{Z} & 0 & \mathbb{Z} \\
\mathrm{BDI} & n=1 & \mathbb{Z}_2^3 & 0 & 0 \\
\mathrm{D} & n=2 & \mathbb{Z}_2^4 & \mathbb{Z}^2 & 0 \\
\mathrm{DIII} & n=3 & 0 & 0 & 0 \\
\mathrm{AII} & n=4 & \mathbb{Z} & \mathbb{Z}_2^3 & \mathbb{Z} \\
\mathrm{CII} & n=5 & 0 & 0 & 0 \\
\mathrm{C} & n=6 & 0 & \mathbb{Z}^2 & \mathbb{Z}_2 \\
\mathrm{CI} & n=7 & 0 & 0 & 0 \\
\hline
 & E_\infty^{p,-n} & p=0 & p=1 & p=2
\end{array} \tag{4.21}$$

The data $\{E_\infty^{0,-n}, E_\infty^{1,-(n+1)}, E_\infty^{2,-(n+2)}\}$ approximate the K-group ${}^\phi K_{\mathbb{Z}_2}^{-n}(T^2)$ on the basis of the exact sequences (4.14). The relationship among columns of E_∞-page, AZ symmetry classes of $2d$ bulk insulators and $2d$ surface gapless states should be kept in mind, which is shown below:

		$p=0$	$p=1$	$p=2$	
AI	$n=0$	$\mathbb{Z}$	0	$\mathbb{Z}$	← C bulk insulators = CI gapless surface states
BDI	$n=1$	$\mathbb{Z}_2^3$	0	0	← CI bulk insulators = AI gapless surface states
D	$n=2$	$\mathbb{Z}_2^4$	$\mathbb{Z}^2$	0	← AI bulk insulators = BDI gapless surface states
DIII	$n=3$	0	0	0	← BDI bulk insulators = D gapless surface states
AII	$n=4$	$\mathbb{Z}$	$\mathbb{Z}_2^3$	$\mathbb{Z}$	← D bulk insulators = DIII gapless surface states
CII	$n=5$	0	0	0	← DIII bulk insulators = AII gapless surface states
C	$n=6$	0	$\mathbb{Z}^2$	$\mathbb{Z}_2$	← AII bulk insulators = CII gapless surface states
CI	$n=7$	0	0	0	← CII bulk insulators = C gapless surface states
	$E_\infty^{p,-n}$	$p=0$	$p=1$	$p=2$	

The dimension p of $E_\infty^{p,-n}$ indicates the dimension of the skeleton X_p on which the topological invariant is defined. The exact sequences (4.14) are recast as

$$\begin{aligned}
&\text{AI bulk insulators}: && {}^\phi K_{\mathbb{Z}_2}^{0}(T^2) = \mathbb{Z}, \\
&\text{BDI bulk insulators}: && 0 \to \mathbb{Z}^2 \to {}^\phi K_{\mathbb{Z}_2}^{-1}(T^2) \to \mathbb{Z}_2^3 \to 0, \\
&\text{D bulk insulators}: && 0 \to \mathbb{Z} \to {}^\phi K_{\mathbb{Z}_2}^{-2}(T^2) \to \mathbb{Z}_2^4 \to 0, \\
&\text{DIII bulk insulators}: && {}^\phi K_{\mathbb{Z}_2}^{-3}(T^2) = \mathbb{Z}_2^3, \\
&\text{AII bulk insulators}: && {}^\phi K_{\mathbb{Z}_2}^{-4}(T^2) = \mathbb{Z} + \mathbb{Z}_2, \\
&\text{CII bulk insulators}: && {}^\phi K_{\mathbb{Z}_2}^{-5}(T^2) = \mathbb{Z}^2, \\
&\text{C bulk insulators}: && {}^\phi K_{\mathbb{Z}_2}^{-6}(T^2) = \mathbb{Z}, \\
&\text{CI bulk insulators}: && {}^\phi K_{\mathbb{Z}_2}^{-7}(T^2) = 0.
\end{aligned}$$

These agree with the literature. [4] Especially, $E_\infty^{2,-6}=\mathbb{Z}_2$ corresponds to the Kane-Mele $\mathbb{Z}_2$ topological invariant. From the explicit formulas of topological invariants, one can show that both the K-groups for $n=1,2$ correspond to the nontrivial extension of the short exact sequence. We find that ${}^\phi K_{\mathbb{Z}_2}^{-1}(T^2)=\mathbb{Z}^2+\mathbb{Z}_2^2$ and ${}^\phi K_{\mathbb{Z}_2}^{-2}(T^2)=\mathbb{Z}+\mathbb{Z}_2^3$. [10]

C. E_1-page for general symmetry

In this section we describe how to compute the E_1-page for general symmetry classes. Let G be the symmetry group and $\phi, c : G \to \mathbb{Z}_2 = \{\pm 1\}$ the indicators for unitary/antiunitary and symmetry/antisymmetry, respectively.

[10] As seen previously, in class BDI bulk insulators, the $1d$ winding numbers w_{1d}^x, w_{1d}^y along the k_x and k_y directions give the constraints on the $\mathbb{Z}_2$ invariants defined at 0-cells as $(-1)^{w_{1d}^x} = (-1)^{\nu(\Gamma)}(-1)^{\nu(X)} = (-1)^{\nu(Y)}(-1)^{\nu(M)}$ and $(-1)^{w_{1d}^y} = (-1)^{\nu(\Gamma)}(-1)^{\nu(Y)} = (-1)^{\nu(X)}(-1)^{\nu(M)}$, which leads to ${}^\phi K_{\mathbb{Z}_2}^{-n}(T^2)=\mathbb{Z}^2+\mathbb{Z}_2^2$. In class D bulk insulators, the parity of the Chern number C is related to the $\mathbb{Z}_2$ invariants (the Pfaffian invariants) at 0-cells as $(-1)^C = \prod_{\boldsymbol{k}\in\Gamma,X,Y,M}(-1)^{\nu(\boldsymbol{k})}$, which leads to ${}^\phi K_{\mathbb{Z}_2}^{-2}(T^2)=\mathbb{Z}+\mathbb{Z}_2^3$.

☆ d_r : "top. phase transition" @ p-cell

and "creating gapless pts" @

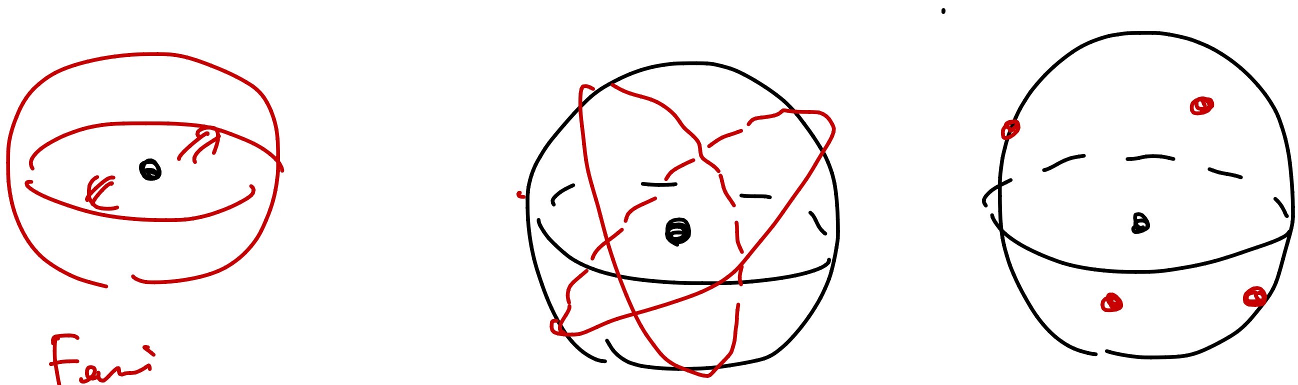
$(p+r)$-cells.

• This process can be modeled by a

<u>Dirac Hamiltonian.</u> !

d_1 $\qquad$ d_2 $\qquad$ d_3

(k_x, k_y, k_z)

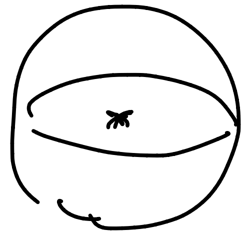


Fermi surface

$H(\mathbf{k}) = (k^2-\mu)\Gamma$

$H(\mathbf{k}) = f(\mathbf{k})\gamma_1 + (k^2-\mu)\Gamma$

$H(\mathbf{k}) = f_1(\mathbf{k})\gamma_1 + f_2(\mathbf{k})\gamma_2 + (k^2-\mu)\Gamma.$

in general, $\mathbb{k} \in D^d$. 

<u>d_{r+1}</u>

$$H_p(\mathbb{k}) = \sum_{\mu=1}^{r} f_\mu(\mathbb{k})\gamma_\mu + (\mathbb{k}^2-\mu)\Gamma.$$

sym : $\hat{g} H(\mathbb{k}) \hat{g}^{-1} = \pm H(O_g \mathbb{k})$

O_g : point group.

$\leadsto \quad \hat{g}\gamma_\mu \hat{g}^{-1} = \gamma_\mu \cdot [\rho(g)]_{\mu\nu}.$

$\rho(g)$: r-dim real rep of G.

$$H(f_1, \cdots, f_r) = \sum_{\mu=1}^{r} f_\mu \gamma_\mu + M\Gamma.$$

for a given r-dim real rep ρ,

one can classify all possible Dirac Hamiltonian. $\Rightarrow$ classification of H_ρ.

Cornfeld - Chapman trick (equivariant Thom iso)

(arXiv:1811.01977, KS. 1907.09354)

$$H(k) = \sum_{\mu=1}^{d} k_\mu \gamma_\mu + m\Gamma.$$

$$\hat{g}\, \gamma\, \hat{g}^{-1} = \gamma \cdot O_g. \quad O_g : \text{point grp.}$$

$$O_g = \begin{cases} R_g \in SO(d) \\ M_1 \cdot R_g \end{cases}$$

reflection

$$\begin{array}{ccc} SO(d) & \longrightarrow & Spin(d) \\ \cup & & \\ R_g & \longrightarrow & U_g. \end{array}$$

$\star$ $$U_g \gamma U_g^{-1} = \gamma \cdot O_g^{-1}.$$

$\therefore$ One can make $\hat{g}$ onsite by

$$\hat{g} \mapsto \tilde{\hat{g}} := U_g \hat{g}.$$

$$\tilde{g}\, H(k)\, \tilde{g}^{-1} = \pm H(k).$$

$\Rightarrow$ $H(k)$ is classified by irreducible chractor.

ex)

$$K_G^{-n}(S^d) \cong \underbrace{K_G^{(\tau,c)-n}(S^d)}_{\text{onsite}} \quad \text{twist from } \sqcup g$$

$$\cong K_G^{(\tau,c)-n-d}(pt).$$

Back up

Part II AHSS

AHSS in general

$h^*(X)$: a generalized cohomology theory.

$$\emptyset \subset X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$$

filteration. X_p : p-skeleton (p-dim sub space)

- $F^p h^n := \mathrm{Ker}[\, h^n(X) \longrightarrow h^n(X_{p-1}) \,]$.

- $E_1^{p,q} := h^{p+q}(X_p, X_{p-1})$.

- differentials

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,\, q-r+1}$$

$\Downarrow$ cohomology

$$E_{r+1}^{p,q} := \mathrm{Ker}\, d_r^{p,q} \,/\, \mathrm{Im}\, d_r^{p-r,\, q+r-1}.$$

- E_∞-page

$$E_1 \Rightarrow E_2 \Rightarrow \cdots \Rightarrow E_\infty. \text{ (if it converges)}$$

- E_∞-page approximate $h^*(X)$:

$$h^n(X) = F^0h^n \supset F^1h^n \supset \cdots \supset F^ph^n \supset F^{p+1}h^n \supset \cdots$$

$$\uparrow$$

$$E_\infty^{p,n-p}.$$

$$E_\infty^{p,n-p} \cong F^ph^n / F^{p+1}h^n.$$

twisted equivariant K-theory

$${}^{\phi}K_{G}^{(\tau,c)-n}(X)$$

- G. finite group $\curvearrowright X$
- $\phi : G \to \mathbb{Z}_2$: $\begin{cases} \text{unitary} \\ \text{anti unitary} \end{cases}$.
- $c : G \to \mathbb{Z}_2$: $\hat{g} H \hat{g}^{-1} = \pm H$.
- τ : "factor system"

$$\hat{g}\hat{h} = e^{i\tau_{g,h}} \widehat{gh}. \quad g,h \in G.$$

($\hat{g}$, $\tau_{g,h}$ depends on $k \in X$).

- $n \in \mathbb{Z}$, grading.

(Bott periodicity $K^{-n} \cong K^{-n+8}$).