Local Unitary Transformations and $H^3(G,U(1))$

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Abstract

This is a calculation memo on the method of Else–Nayak [1] and on a concrete example in the Levin–Gu model [2].

1 The Else–Nayak method

In general, in a nonchiral invertible phase, or SPT phase, it is believed that the symmetry operator realized in the low-energy Hilbert space on the boundary of the system cannot be written as a product of on-site unitaries,

$$\bigotimes_x u_x(g), \tag{1}$$

but can be written as a local unitary transformation, namely as the finite-time evolution generated by a local Hamiltonian. Focusing on this point, Ref. [1] shows that the classification of SPT phases is given by the group cohomology $H^{d+1}(G,U(1))$. I first summarize the method of Ref. [1] in two spatial dimensions, and then check it in the concrete example of Ref. [2].

In an SPT phase, let $U_{S^1}(g\in G)$ be the symmetry operator projected to the low-energy Hilbert space on the boundary $S^1=\partial D^2$ of a two-dimensional disk. There is no ambiguity in $U_{S^1}(g)$. Take an interval $I=[a,b]$ of the boundary circle S^1, restrict $U_{S^1}(g)$ to I, and denote the result by $U_I(g)$.

- Since $U_{S^1}(g)$ can be written as a local unitary transformation, one can obtain $U_I(g)$ by restricting it in a suitable way.
- On the other hand, $U_I(g)$ has an ambiguity by local unitary transformations supported near the boundary of I.

Because of the ambiguity at the boundary ∂I[1], the group law holds only up to local unitaries at ∂I:

$$U_I(g_1)U_I(g_2)=\Omega_{\partial I}(g_1,g_2)U_I(g_1g_2). \tag{2}$$

Here $\Omega_{\partial I}(g_1,g_2)$ denotes a local unitary transformation supported near ∂I.

- Thus the classification problem for the local unitary transformations $U_{S^1}(g)$ is reduced to the classification of the "obstruction" local unitaries $\Omega_{\partial I}(g_1,g_2)$.

The operators $\Omega_{\partial I}$ are not arbitrary. Associativity, $(U_I(g_1)U_I(g_2))U_I(g_3)=U_I(g_1)(U_I(g_2)U_I(g_3))$, imposes the constraint

$$\Omega_{\partial I}(g_1,g_2)\Omega_{\partial I}(g_1g_2,g_3)={}^{U_I(g_1)}\Omega_{\partial I}(g_2,g_3)\Omega_{\partial I}(g_1,g_2g_3). \tag{3}$$

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[1] This should be defined carefully.

Here I introduced the left action

$$ {}^{U_I(g_1)}\Omega_{\partial I}(g_2,g_3) = U_I(g_1)\Omega_{\partial I}(g_2,g_3)U_I(g_1)^{-1}. \tag{4}$$

Next restrict $\Omega_{\partial I}(g_1,g_2)$ to one boundary point a, and denote by $\Omega_a(g_1,g_2)$ the local unitary transformation supported near a. Since a $U(1)$ phase ambiguity appears, one can write

$$\Omega_a(g_1,g_2)\Omega_a(g_1g_2,g_3) = \omega(g_1,g_2,g_3)^{U_I(g_1)}\Omega_a(g_2,g_3)\Omega_a(g_1,g_2g_3), \tag{5}$$

where $\omega(g_1,g_2,g_3)$ is a $U(1)$ phase. One can show that $\omega(g_1,g_2,g_3)$ satisfies the cocycle condition

$$\delta\omega = 0. \tag{6}$$

2 The Levin–Gu model

Write $\mathbb{Z}_2 = \{e,\sigma\}$. In the Levin–Gu model [2], the $\mathbb{Z}_2$ symmetry transformation on the one-dimensional edge is

$$U_{S^1}(e) = \mathrm{Id}, \tag{7}$$

$$U_{S^1}(\sigma) = \prod_j \sigma^x_j \prod_j e^{\frac{\pi i}{2}\frac{1-\sigma^z_j\sigma^z_{j+1}}{2}}. \tag{8}$$

Notice that the second factor has an ambiguity on a finite system. For example, on a closed chain,

$$\prod_j e^{\frac{\pi i}{2}\frac{1-\sigma^z_j\sigma^z_{j+1}}{2}} = \prod_j e^{\pi i\frac{1+\sigma^z_j}{2}\frac{1-\sigma^z_{j+1}}{2}} \tag{9}$$

gives the same local unitary, but the two expressions differ on an interval $I=[1,L]$. Let us take the local unitary transformation on the finite interval $I=[1,L]$ to be

$$U_I(e) = \mathrm{Id}, \tag{10}$$

$$U_I(\sigma) = \prod_{j=1}^{L}\sigma^x_j \prod_{j=1}^{L-1} e^{\frac{\pi i}{2}\frac{1-\sigma^z_j\sigma^z_{j+1}}{2}}. \tag{11}$$

Then

$$\begin{aligned}
U_I(\sigma)^2 &= \prod_{j=1}^{L}\sigma^x_j \prod_{j=1}^{L-1} e^{\frac{\pi i}{2}\frac{1-\sigma^z_j\sigma^z_{j+1}}{2}} \prod_{j=1}^{L}\sigma^x_j \prod_{j=1}^{L-1} e^{\frac{\pi i}{2}\frac{1-\sigma^z_j\sigma^z_{j+1}}{2}} && (12)\\
&= \prod_{j=1}^{L-1}(-1)^{\frac{1-\sigma^z_j\sigma^z_{j+1}}{2}} && (13)\\
&= \prod_{j=1}^{L-1}\sigma^z_j\sigma^z_{j+1} && (14)\\
&= \sigma^z_1\sigma^z_L. && (15)
\end{aligned}$$

Therefore

$$\Omega_{\partial I}(g_1,g_2) = \begin{cases} \sigma^z_1\sigma^z_L & (g_1=g_2=\sigma), \\ \mathrm{Id} & (\text{else}), \end{cases} \tag{16}$$

is obtained. Note that

$$\begin{aligned}
{}^{U_I(\sigma)}\Omega_{\partial I}(\sigma,\sigma) &= \Big(\prod_{j=1}^{L}\sigma^x_j \prod_{j=1}^{L-1} e^{\frac{\pi i}{2}\frac{1-\sigma^z_j\sigma^z_{j+1}}{2}}\Big)(\sigma^z_1\sigma^z_L)\Big(\prod_{j=1}^{L}\sigma^x_j \prod_{j=1}^{L-1} e^{\frac{\pi i}{2}\frac{1-\sigma^z_j\sigma^z_{j+1}}{2}}\Big)^{-1} && (17)\\
&= (-\sigma^z_1)(-\sigma^z_L). && (18)
\end{aligned}$$

The restriction to the left endpoint of the interval I can be chosen as

$$\Omega_a(g_1,g_2)=\left\{\begin{array}{ll}\sigma_1^z & (g_1=g_2=\sigma),\\ \mathbf{1} & (\text{else}).\end{array}\right. \tag{19}$$

The only nontrivial left action is

$$\begin{aligned} {}^{U_I(\sigma)}\Omega_a(\sigma,\sigma) &= U_I(\sigma)\Omega_a(\sigma,\sigma)U_I(\sigma)^{-1} &(20)\\ &= (\prod_{j=1}^{L}\sigma_j^x\prod_{j=1}^{L-1}e^{\frac{\pi i}{2}\frac{1-\sigma_j^z\sigma_{j+1}^z}{2}})(\sigma_1^z)(\prod_{j=1}^{L}\sigma_j^x\prod_{j=1}^{L-1}e^{\frac{\pi i}{2}\frac{1-\sigma_j^z\sigma_{j+1}^z}{2}})^{-1} &(21)\\ &= -\sigma_1^z. &(22)\end{aligned}$$

From this one sees that, among the values of the three-cocycle ω, the only value not equal to 1 occurs for $(g_1,g_2,g_3)=(\sigma,\sigma,\sigma)$. Thus

$$\omega(g_1,g_2,g_3)=\left\{\begin{array}{ll}-1 & (g_1=g_2=g_3=\sigma),\\ 1 & (\text{else})\end{array}\right. \tag{23}$$

is obtained. It is known that such an ω is a nontrivial group cocycle in $H^3(\mathbb{Z}_2,U(1))$.

Let us also check another choice of $U_I(\sigma)$. If

$$\tilde{U}_I(\sigma)=\prod_{j=1}^{L}\sigma_j^x\prod_{j=1}^{L-1}e^{\pi i\frac{1+\sigma_j^z}{2}\frac{1-\sigma_{j+1}^z}{2}}, \tag{24}$$

then $\tilde{U}_I(\sigma)$ and $U_I(\sigma)$ differ only by local unitary transformations at the endpoints of I. Indeed,

$$\begin{aligned}\tilde{U}_I(\sigma) &= \prod_{j=1}^{L}\sigma_j^x(i\sigma_1^z)\left(\prod_{j=1}^{L-1}e^{\frac{\pi i}{2}\frac{1-\sigma_j^z\sigma_{j+1}^z}{2}}\right)(-i\sigma_L^z) &(25)\\ &= \sigma_1^z U_I(\sigma)\sigma_L^z. &(26)\end{aligned}$$

Therefore

$$\tilde{U}_I(\sigma)^2=(\sigma_1^zU_I(\sigma)\sigma_L^z)(\sigma_1^zU_I(\sigma)\sigma_L^z)=U_I(\sigma)^2, \tag{27}$$

so ω is unchanged.

Finally, let us consider the relation

$$\Omega_a(\sigma,\sigma)\Omega_a(e,\sigma)=-{}^{U_I(\sigma)}\Omega_a(\sigma,\sigma)\Omega_a(\sigma,e). \tag{28}$$

The origin of this relation is the dependence on the order in which the $\mathbb{Z}_2$ symmetry transformation $U_I(\sigma)U_I(\sigma)U_I(\sigma)$ on the interval I is decomposed:

$$(U_I(\sigma)U_I(\sigma))U_I(\sigma)=\Omega_{\partial I}(\sigma,\sigma)U_I(e)U_I(\sigma)=\Omega_{\partial I}(\sigma,\sigma)\Omega_a(e,\sigma)U_I(\sigma), \tag{29}$$

$$\begin{aligned}U_I(\sigma)(U_I(\sigma)U_I(\sigma)) &= U_I(\sigma)\Omega_{\partial I}(\sigma,\sigma)U_I(e) &(30)\\ &= {}^{U_I(\sigma)}\Omega_{\partial I}(\sigma,\sigma)\Omega_a(\sigma,e)U_I(\sigma). &(31)\end{aligned}$$

Drawn pictorially, this gives Fig. 1.

References

[1] Dominic V. Else and Chetan Nayak, *Classifying symmetry-protected topological phases through the anomalous action of the symmetry on the edge*, arXiv:1409.5436.

[2] Michael Levin and Zheng-Cheng Gu, *Braiding statistics approach to symmetry-protected topological phases*, arXiv:1202.3120.

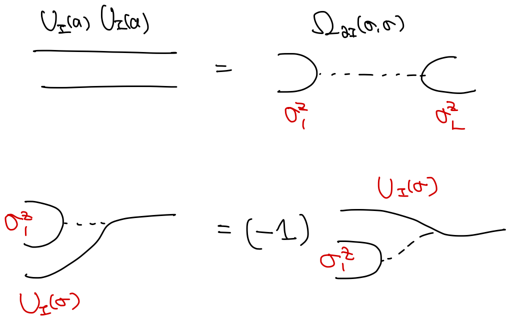

Figure 1