# Note: Two-Parameter Pumps

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## 1 Can a model of a two-parameter pump be constructed by applying the Bockstein map twice?

We first recall the construction of a one-parameter pump. Let

$$\omega^{(d)} \in Z^d(G, \mathbb{R}/\mathbb{Z}) \tag{1}$$

be given. Choose a lift

$$\omega^{(d)} \mapsto \widetilde{\omega}^{(d)} \in \mathbb{R}. \tag{2}$$

Then note that

$$d\widetilde{\omega}^{(d)} \in Z^{d+1}(G, \mathbb{Z}). \tag{3}$$

A $(d+1)$-cocycle with coefficients in $\mathbb{R}/\mathbb{Z}$, parametrized by $t \in \mathbb{R}/\mathbb{Z}$, is obtained as

$$\omega_t^{(d+1)} = t\, d\widetilde{\omega}^{(d)}. \tag{4}$$

Naively, one might think that a model of a two-parameter pump can be obtained by applying the above Bockstein map once more to $\omega_t^{(d+1)}$. However, this seems not to work because one cannot construct a lift that is continuous in $t \in \mathbb{R}/\mathbb{Z}$. Let us examine this point with an example.

Let

$$G = \mathbb{Z}_2 = \{e, \sigma\}. \tag{5}$$

Take $\omega^{(1)} \in Z^1(\mathbb{Z}_2, \mathbb{R}/\mathbb{Z})$ to be

$$\omega^{(1)}(e) = 0, \quad \omega^{(1)}(\sigma) = 1/2. \tag{6}$$

Choose the lift with the same representatives:

$$\widetilde{\omega}^{(1)}(e) = 0, \quad \widetilde{\omega}^{(1)}(\sigma) = 1/2. \tag{7}$$

Then

$$(d\widetilde{\omega}^{(1)})(g,h) = \begin{cases} 1 & (g = h = \sigma), \\ 0 & (\text{otherwise}). \end{cases} \tag{8}$$

Therefore the one-parameter family of 2-cocycles

$$\omega_t^{(2)} \in Z^2(\mathbb{Z}_2, \mathbb{R}/\mathbb{Z}) \tag{9}$$

is

$$\omega_t^{(2)}(g,h) = \begin{cases} t & (g = h = \sigma), \\ 0 & (\text{otherwise}). \end{cases} \tag{10}$$

We would next like to lift $\omega_t^{(2)}$ to an $\mathbb{R}$-valued cochain. However, it is impossible to lift $\omega_t^{(2)}(\sigma,\sigma) = t$ to an $\mathbb{R}$-valued function while preserving continuity in $t \in \mathbb{R}/\mathbb{Z}$.

This difficulty is not in contradiction with the fact that, as a space,

$$Z^3(\mathbb{Z}_2, \mathbb{R}/\mathbb{Z}) \cong (\mathbb{R}/\mathbb{Z})^2. \tag{11}$$

Since $(\mathbb{R}/\mathbb{Z})^2$ has trivial $\pi_2$, a cocycle model based on elements of $Z^3(\mathbb{Z}_2, \mathbb{R}/\mathbb{Z})$ cannot produce a nontrivial two-parameter pump.