# The Baer Sum

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**Abstract**

## 1 The Baer sum

Let

$$0 \to B \xrightarrow{f_0} E_0 \xrightarrow{g_0} A \to 0, \tag{1}$$
$$0 \to B \xrightarrow{f_1} E_1 \xrightarrow{g_1} A \to 0 \tag{2}$$

be extensions of $\mathbb{Z}$-modules. We would like to construct the extension corresponding to the sum in $\mathrm{Ext}^1_{\mathbb{Z}}(A, B)$. Define

$$f : B \to E_0 \oplus E_1, \quad b \mapsto (f_0(b), -f_1(b)), \tag{3}$$
$$g : E_0 \oplus E_1 \to A, \quad (e_0, e_1) \mapsto g_0(e_0) - g_1(e_1). \tag{4}$$

The inclusion $\mathrm{Im}\, f \subset \mathrm{Ker}\, g$ follows from

$$b \mapsto (f_0(b), -f_1(b)) \mapsto g_0 \circ f_0(b) + g_1 \circ f_1(b) = 0 + 0. \tag{5}$$

The Baer sum is given by

$$0 \to B \xrightarrow{f'} \mathrm{Ker}\, g/\mathrm{Im}\, f \xrightarrow{g'} A \to 0. \tag{6}$$

Here

$$f' : B \to \mathrm{Ker}\, g/\mathrm{Im}\, f, \quad b \mapsto [(f_0(b), 0)] = [(0, f_1(b))], \tag{7}$$
$$g' : \mathrm{Ker}\, g/\mathrm{Im}\, f \to A, \quad (e_0, e_1) \mapsto g_0(e_0) = g_1(e_1). \tag{8}$$

Suppose $(f_0(b), 0) = (f_0(b'), -f_1(b'))$. Then $(f_0(b - b'), f_1(b')) = (0, 0)$, and by injectivity of $f_0$ and $f_1$ one gets $b = b' = 0$. Thus note that the image of $B \to \mathrm{Ker}\, g$, $b \mapsto (f_0(b), 0)$, has no intersection with $\mathrm{Im}\, f$. The map $g'$ is well-defined because $(f_0(b), -f_1(b)) \mapsto g_0(f_0(b)) = 0 = g_1(f_1(b))$.

Let us prove exactness. For injectivity of $f'$, assume $[(f_0(b), 0)] = 0$. Then there exists $b' \in B$ such that $(f_0(b), 0) = (f_0(b'), -f_1(b'))$, and the same argument gives $b = 0$. Surjectivity of $g'$ follows from the surjectivity of $g_0$ and $g_1$. The inclusion $\mathrm{Im}\, f' \subset \mathrm{Ker}\, g'$ follows from $[f_0(b), 0] \mapsto g_0(f_0(b)) = 0$. We show $\mathrm{Ker}\, g' \subset \mathrm{Im}\, f'$. Assume $[(e_0, e_1)] \mapsto g_0(e_0) = g_1(e_1) = 0$. By exactness of the original sequences, there exist $b, b' \in B$ such that $e_0 = f_0(b)$ and $e_1 = f_1(b')$. Then

$$[(e_0, e_1)] = [(f_0(b), f_1(b'))] = [(f_0(b - b'), 0)] \in \mathrm{Im}\, f'. \tag{9}$$

If one is interested only in the $\mathbb{Z}$-module obtained as the result of the Baer sum of extensions, it suffices to compute $\mathrm{Ker}\, g/\mathrm{Im}\, f$. If one subsequently takes another Baer sum with the extension obtained here, then one also needs to compute $f'$ and $g'$. Let us compute by hand how $f'$ and $g'$ are obtained in several examples.

## 1.1 Example 1

Compute the Baer sum of the extension

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}_4 & \longrightarrow & 0 \\
 & & 1 & \longrightarrow & 4 & & & & \\
 & & & & 1 & \longrightarrow & 1 & &
\end{array} \tag{10}
$$

with itself.

$$
\begin{array}{ccccc}
\mathbb{Z} & \xrightarrow{f} & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{g} & \mathbb{Z}_4 \\
1 & \longrightarrow & (4,-4) & & \\
 & & (n,m) & \longrightarrow & n-m
\end{array} \tag{11}
$$

Then

$$
\operatorname{Im} f = \mathbb{Z}(4,-4), \tag{12}
$$

$$
\operatorname{Ker} g = \mathbb{Z}(1,1)\oplus\mathbb{Z}(4,0) \cong \mathbb{Z}(1,1)\oplus\mathbb{Z}(2,-2), \tag{13}
$$

and hence

$$
\operatorname{Ker} g/\operatorname{Im} f = \mathbb{Z}(1,1)\oplus\mathbb{Z}_2(2,-2). \tag{14}
$$

Next determine $f'$ and $g'$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{f'} & \frac{\mathbb{Z}(1,1)\oplus\mathbb{Z}(2,-2)}{\mathbb{Z}(4,-4)} & \xrightarrow{g'} & \mathbb{Z}_4 & \longrightarrow & 0 \\
 & & 1 & \longrightarrow & [(4,0)] = [(2,2)] + [(2,-2)] & & & & \\
 & & & & (1,1) & \longrightarrow & 1 & & \\
 & & & & (2,-2) & \longrightarrow & 2 & &
\end{array} \tag{15}
$$

Thus

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{f'} & \mathbb{Z}\oplus\mathbb{Z}_2 & \xrightarrow{g'} & \mathbb{Z}_4 & \longrightarrow & 0 \\
 & & 1 & \longrightarrow & (2,1) & & & & \\
 & & & & (1,0) & \longrightarrow & 1 & & \\
 & & & & (0,1) & \longrightarrow & 2 & &
\end{array} \tag{16}
$$

Let us carry out the above calculation as mechanically as possible. First, change the basis of $\mathbb{Z}\oplus\mathbb{Z}$ so that $\operatorname{Im} f$ becomes a direct-sum component of $\operatorname{Ker} g$. Expanding $\operatorname{Im} f$ in the basis of $\operatorname{Ker} g$, the expansion coefficients $(x, y)$ are

$$
(4,-4)\begin{pmatrix} 1,1 \\ 4,0 \end{pmatrix}^{+} = (-4,2). \tag{17}
$$

Taking the Smith decomposition,

$$
u(-4,2)v = (2,0), \quad v = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \tag{18}
$$

one finds $\operatorname{Ker} g/\operatorname{Im} f = \mathbb{Z}_2\oplus\mathbb{Z}$. The basis of $\operatorname{Ker} g$ is

$$
v^{-1}\begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 3 & -1 \end{pmatrix}. \tag{19}
$$

The map $f'$ is

$$f' : 1 \mapsto (4, 0). \tag{20}$$

Expanding this in the basis $\mathbb{Z}(-2, 2) \oplus \mathbb{Z}(3, -1)$ gives

$$(4, 0) \begin{pmatrix} -2 & 2 \\ 3 & -1 \end{pmatrix}^{+} = (4, 0) \begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix}^{+} v = (1, 2), \tag{21}$$

so these are the components of $f'(1)$ in the new basis. For $g'$, it is determined as

$$g' : \begin{pmatrix} -2 & 2 \\ 3 & -1 \end{pmatrix} \mapsto \begin{pmatrix} g_0(-2) \\ g_0(3) \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}. \tag{22}$$

## 1.2 Computational implementation 1: the case where $B, E_0, E_1$ are all free $\mathbb{Z}$-modules

We generalize the computation in the example above.

$$\begin{array}{ccccc} & & & & P_A \\ & & & & \downarrow \\ & & E_0 \oplus E_1 & \xrightarrow{\tilde{g}} & \tilde{A} \\ & & \downarrow \cong & & \downarrow \\ B & \xrightarrow{f} & E_0 \oplus E_1 & \xrightarrow{g} & A \end{array} \tag{23}$$

Here $A$ is a torsion $\mathbb{Z}$-module.

- Compute
$$\operatorname{Ker} g = \operatorname{Ker} (\tilde{g} \oplus \operatorname{Id}_{P_A})|_{E_0 \oplus E_1}.$$
- Expand $\operatorname{Im} f$ in the basis of $\operatorname{Ker} g$. Let the expansion matrix be $M$. Writing, by abuse of notation, the embeddings of $\operatorname{Im} f$ and $\operatorname{Ker} g$ into $E_0 \oplus E_1$ again as $\operatorname{Im} f$ and $\operatorname{Ker} g$, we have
$$M = (\operatorname{Im} f)(\operatorname{Ker} g)^{+}.$$
- Compute the Smith decomposition $uMv = \Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, 0, \ldots)$. The entries $(\lambda_1, \lambda_2, \ldots, 0, \ldots)$ give the structure of $\operatorname{Ker} g / \operatorname{Im} f$ as a $\mathbb{Z}$-module. That is, $\lambda_p$ corresponds to $\mathbb{Z}_{\lambda_p}$, while 0 corresponds to $\mathbb{Z}$.
- The expression for $f'$ in the basis transformed by $v$ is as follows. Writing the embedding $\operatorname{Im} f_0 \subset E_0 \subset E_0 \oplus E_1$ as $\operatorname{Im} f_0$ itself, it is given by
$$(\operatorname{Im} f_0)(\operatorname{Ker} g)^{+} v.$$
- For $g'$, look at the image of $\operatorname{Ker} g$ in the basis transformed by $v$. In other words, project the matrix $v^{-1}(\operatorname{Ker} g)$ to $E_0$ and then map it by $g_0$. Since the matrix representation of $\operatorname{Im} g_0$ is in the standard basis of $E_0$, the matrix representation of $g'$ is given by
$$v^{-1}(\operatorname{Ker} g)(\operatorname{Im} g_0).$$

### 1.3 Example 2

As a slightly more complicated example, compute the Baer sum of the following two extensions.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z}_2 & \xrightarrow{f_0} & \mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2 & \xrightarrow{g_0} & \mathbb{Z}_4 & \longrightarrow & 0 \\
 & & (1,0) & \longrightarrow & (2,1,0) & & & & \\
 & & (0,1) & \longrightarrow & (0,0,1) & & & & \\
 & & & & (1,0,0) & \longrightarrow & 1 & & \\
 & & & & (0,1,0) & \longrightarrow & 2 & & \\
 & & & & (0,0,1) & \longrightarrow & 0 & &
\end{array} \tag{24}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z}_2 & \xrightarrow{f_1} & \mathbb{Z}\oplus\mathbb{Z}_8 & \xrightarrow{g_1} & \mathbb{Z}_4 & \longrightarrow & 0 \\
 & & (1,0) & \longrightarrow & (1,0) & & & & \\
 & & (0,1) & \longrightarrow & (0,4) & & & & \\
 & & & & (1,0) & \longrightarrow & 0 & & \\
 & & & & (0,1) & \longrightarrow & 1 & &
\end{array} \tag{25}$$

In this case,

$$\begin{array}{ccccc}
\mathbb{Z}\oplus\mathbb{Z}_2 & \xrightarrow{f} & \mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}\oplus\mathbb{Z}_8 & \xrightarrow{g} & \mathbb{Z}_4 \\
(1,0) & \longrightarrow & (2,1,0,-1,0) & & \\
(0,1) & \longrightarrow & (0,0,1,0,-4) & & \\
 & & (n,m,k,p,q) & \longrightarrow & n+2m-q
\end{array} \tag{26}$$

Therefore

$$\operatorname{Im} f = \mathbb{Z}(2,1,0,-1,0)\oplus\mathbb{Z}_2(0,0,1,0,-4), \tag{27}$$
$$\operatorname{Ker} g = \mathbb{Z}(1,0,0,0,1)\oplus\mathbb{Z}_4(0,1,0,0,2)\oplus\mathbb{Z}_2(0,0,1,0,0)\oplus\mathbb{Z}(0,0,0,1,0) \tag{28}$$
$$\cong \mathbb{Z}(1,0,0,0,1)\oplus\mathbb{Z}_4(0,1,0,0,2)\oplus\mathbb{Z}_2(0,0,1,0,-4)\oplus\mathbb{Z}(-2,-1,0,1,0). \tag{29}$$

From this,

$$\operatorname{Ker} g/\operatorname{Im} f = \mathbb{Z}[(1,0,0,0,1)]\oplus\mathbb{Z}_4[(0,1,0,0,2)]. \tag{30}$$

Let us compute $f'$ and $g'$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z}_2 & \xrightarrow{f'} & \frac{\mathbb{Z}(1,0,0,0,1)\oplus\mathbb{Z}_4(0,1,0,0,2)\oplus\mathbb{Z}_2(0,0,1,0,-4)\oplus\mathbb{Z}(-2,-1,0,1,0)}{\mathbb{Z}(2,1,0,-1,0)\oplus\mathbb{Z}_2(0,0,1,0,-4)} & \xrightarrow{g'} & \mathbb{Z}_4 & \longrightarrow & 0 \\
 & & (1,0) & \longrightarrow & [(2,1,0,0,0)] = 2[(1,0,0,0,1)] - [(0,1,0,0,2)] & & & & \\
 & & (0,1) & \longrightarrow & [(0,0,1,0,0)] = 2[(0,1,0,0,2)] & & & & \\
 & & & & (1,0,0,0,1) & \longrightarrow & 1 & & \\
 & & & & (0,1,0,0,2) & \longrightarrow & 2 & &
\end{array} \tag{31}$$

Thus

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z}_2 & \xrightarrow{f'} & \mathbb{Z}\oplus\mathbb{Z}_4 & \xrightarrow{g'} & \mathbb{Z}_4 & \longrightarrow & 0 \\
 & & (1,0) & \longrightarrow & (2,-1) & & & & \\
 & & (0,1) & \longrightarrow & (0,2) & & & & \\
 & & & & (1,0) & \longrightarrow & 1 & & \\
 & & & & (0,1) & \longrightarrow & 2 & &
\end{array}
\tag{32}
$$

## 1.4 Computational implementation 2: the general case

$$
\begin{array}{ccccc}
P_B & & P_{E_0}\oplus P_{E_1} & & P_A \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{B} & \xrightarrow{\tilde{f}} & \tilde{E}_0\oplus\tilde{E}_1 & \xrightarrow{\tilde{g}} & \tilde{A} \\
\downarrow & & \downarrow & & \downarrow \\
B & \xrightarrow{f} & E_0\oplus E_1 & \xrightarrow{g} & A
\end{array}
\tag{33}
$$

Here $A$ is a torsion $\mathbb{Z}$-module.

- By the general theory,

$$\mathrm{Ker}\, g = \frac{\mathrm{Ker}\,(\tilde{g}\oplus \mathrm{Id}_{P_A})|_{\tilde{E}_0\oplus\tilde{E}_1}}{\mathrm{Im}\,\tilde{f} + (P_{E_0}\oplus P_{E_1})}. \tag{34}$$

- Let $M_g$ denote the matrix obtained by putting the basis vectors of $\mathrm{Ker}\,(\tilde{g}\oplus \mathrm{Id}_{P_A})|_{\tilde{E}_0\oplus\tilde{E}_1}$ as rows:

$$\mathrm{Ker}\,(\tilde{g}\oplus \mathrm{Id}_{P_A})|_{\tilde{E}_0\oplus\tilde{E}_1} = \left\langle \begin{pmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix} \right\rangle = \langle M_g \rangle\,. \tag{35}$$

  The matrix $M_g$ has size $|\mathrm{Ker}\, g| \times |E_0\oplus E_1|$.

- Similarly, write

$$\mathrm{Im}\,\tilde{f} + (P_{E_0}\oplus P_{E_1}) = \langle M_f \rangle\,. \tag{36}$$

  The matrix $M_f$ need not be chosen to have linearly independent rows. It has size $|B| + |P_{E_0}\oplus P_{E_1}| \times |E_0\oplus E_1|$.

- Expand $\mathrm{Im}\,\tilde{f} + P_{E_0}\oplus P_{E_1}$ in the basis of $\mathrm{Ker}\,(\tilde{g}\oplus \mathrm{Id}_{P_A})|_{\tilde{E}_0\oplus\tilde{E}_1}$. The expansion coefficients are given by

$$M_f M_g^+\,. \tag{37}$$

- Take the Smith decomposition of the matrix $M_f M_g^+$:

$$u(M_f M_g^+)v = \Lambda = \left[\begin{array}{cccccc} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \\ \hline & & & O & & \end{array}\right], \quad \lambda_i|\lambda_{i+1}. \tag{38}$$

The diagonal entries of $\Lambda$ give the $\mathbb{Z}$-module data of $\mathrm{Ker}\, g/\mathrm{Im}\, f$. The entry $\lambda_i$ corresponds to $\mathbb{Z}/\lambda_i\mathbb{Z}$, and 0 corresponds to $\mathbb{Z}$. The matrix

$$v^{-1}M_g = \begin{pmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix} \tag{39}$$

is the basis in which we want to express $f'$ and $g'$. From the top, the rows correspond to $\mathbb{Z}/\lambda_1\mathbb{Z}, \mathbb{Z}/\lambda_2\mathbb{Z}, \ldots, \mathbb{Z}, \ldots$. Note that some $\lambda_i$ may be equal to 1.

- To express $f' : b \mapsto [(f_0(b), 0)]$, first write

  $$f_0(B) = \langle M_{f_0} \rangle \,. \tag{40}$$

  The matrix $M_{f_0}$ has size $|B| \times |E_0|$. Append zeros in the $E_1$ part, and expand the resulting matrix in the basis $v^{-1}M_g$:

  $$(M_{f_0}, O)(v^{-1}M_g)^+ = (M_{f_0}, O)M_g^+ v. \tag{41}$$

  This gives the representation of $f'$.

- Next compute the representation of $g' : (e_0, e_1) \mapsto g_0(e_0)$. Project $\mathrm{Ker}\, g$ to $E_0$:

  $$(v^{-1}M_g)|_{E_0}. \tag{42}$$

  This is a $|\mathrm{Ker}\, g| \times |E_0|$ matrix. Map it by $g_0$. Write

  $$g_0(E_0) = \langle M_{g_0} \rangle \,, \tag{43}$$

  where this is a $|E_0| \times |A|$ matrix. Then the representation of $g'$ is

  $$(v^{-1}M_g)|_{E_0} M_{g_0}. \tag{44}$$

For the Mathematica implementation of the calculation, see [1].

# References


[1] BaerSum.nb.