Memo: The Barycentric Subdivision Operator

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1 The barycentric subdivision operator

I record notes from Chapter 2 of [1].

Let Δ^n be the standard n-simplex. A map

$$\sigma : \Delta^n \to \mathbb{R}^q \tag{1.1}$$

is affine if

$$\sigma((1-t)x + ty) = (1-t)f(x) + tf(y), \quad x, y \in \Delta^n, \quad t \in [0,1] \tag{1.2}$$

holds. Hence an affine map σ is determined by specifying only the images of the vertices of Δ^n. We write, for example,

$$\sigma = (v_0 \cdots v_n), \quad v_i \in \mathbb{R}^q, \quad i = 0, \dots, n. \tag{1.3}$$

[1] The target $\mathbb{R}^q$ may be replaced by any convex space.

Fix Δ^q. Denote by $S'_n(\Delta^q)$ the free module generated by all affine simplices $\sigma : \Delta^n \to \Delta^q$, that is, by all affine n-simplices. Then $S'_*(\Delta^q)$ is a subcomplex of $S_*(\Delta^q)$.

For an affine n-simplex $\sigma = (v_0 \cdots v_n)$, write its barycenter as

$$b_\sigma := \frac{1}{n+1} \sum_{i=0}^{n} v_i \in \Delta^q. \tag{1.4}$$

For this fixed affine simplex σ, define

$$\beta_\sigma : S'_l(\Delta^q) \to S'_{l+1}(\Delta^q), \quad \beta_\sigma(w_0 \cdots w_l) := (b_\sigma w_0 \cdots w_l), \tag{1.5}$$

$$\eta_\sigma : \mathbb{Z} \to S'_*(\Delta^q), \quad \eta_\sigma(k) = k b_\sigma \in S'_0(\Delta^q). \tag{1.6}$$

The map β_σ adds the barycenter of σ to an affine l-simplex $(w_0 \cdots w_l)$, and η_σ sends an integer k to k copies of the barycenter b_σ of σ. Introduce the augmentation map

$$\epsilon : S'_*(\Delta^q) \to \mathbb{Z}, \quad \epsilon(\rho^0 = \sum_i a_i w_i) = 1, \quad \epsilon(\rho^{n>0}) = 0. \tag{1.7}$$

Since $S'_0(\Delta^q)$ is finitely generated in each element under consideration, the sum $\sum_i a_i$ is finite. The composite

$$\eta_\sigma \circ \epsilon : S'_*(\Delta^q) \to S'_*(\Delta^q) \tag{1.8}$$

acts nontrivially only in degree 0 and does not change degree:

$$\eta_\sigma \circ \epsilon(\sum_i a_i w_i) = \eta_\sigma(\sum_i a_i) = (\sum_i a_i) b_\sigma. \tag{1.9}$$

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[1] Notice that a general singular n-simplex $\sigma : \Delta^n \to X$ is an arbitrary continuous map, and X need not be a vector space, so notation such as $\sigma = (v_0 \cdots v_n)$ is meaningless in that generality.

- $\eta_\sigma \circ \epsilon$ is a chain map.

Indeed,

$$\eta_\sigma \circ \epsilon \circ \partial(\rho^1 = (w_0 w_1)) = \eta_\sigma \circ \epsilon(w_1 - w_0) = 0 = \partial \circ \eta_\sigma \circ \epsilon(w_0 w_1), \tag{1.10}$$

$$\partial \circ \eta_\sigma \circ \epsilon(w_0) = \partial \circ \eta_\sigma(1) = \partial(b_\sigma) = 0 = \eta_\sigma \circ \epsilon \circ \partial(\rho^0 = w_0), \tag{1.11}$$

$$\eta_\sigma \circ \epsilon \circ \partial(\rho^{n>1}) = \partial \circ \eta_\sigma \circ \epsilon(\rho^{n>1}) = 0. \tag{1.12}$$

Lemma 1.1. *The map β_σ is a chain homotopy between $\eta_\sigma \circ \epsilon$ and 1; that is,*

$$\partial\beta_\sigma + \beta_\sigma\partial = id - \eta_\sigma\epsilon. \tag{1.13}$$

Proof. This is a direct computation:

$$\partial\beta_\sigma(w_0 \cdots w_l) = (w_0 \cdots w_l) - \sum_{i=0}^{l}(-1)^i(b_\sigma w_0 \cdots \hat{w}_i \cdots w_l), \tag{1.14}$$

$$\beta_\sigma\partial(\rho^0) = 0, \tag{1.15}$$

$$\beta_\sigma\partial(\rho^{l>0} = (w_0 \cdots w_l)) = \beta_\sigma\left(\sum_{i=0}^{l}(-1)^i(w_0 \cdots \hat{w}_i \cdots w_l)\right) \tag{1.16}$$

$$= \sum_{i=0}^{l}(-1)^i(b_\sigma w_0 \cdots \hat{w}_i \cdots w_l). \tag{1.17}$$

Therefore

$$(\partial\beta_\sigma + \beta_\sigma\partial)(\rho^0 = w_0) = (w_0) - (b_\sigma) = (id - \eta_\sigma\epsilon)(w_0), \tag{1.18}$$

$$(\partial\beta_\sigma + \beta_\sigma\partial)(\rho^{l>0} = (w_0 \cdots w_l)) = (w_0 \cdots w_l) = \rho^l = (id - \eta_\sigma\epsilon)(\rho^l). \quad \square \tag{1.19}$$

Define homomorphisms independent of any fixed auxiliary simplex σ,

$$sd_n : S'_n(\Delta^q) \to S'_n(\Delta^q), \quad \sigma = (v_0 \ldots v_n) \mapsto sd_n(v_0 \cdots v_n), \tag{1.20}$$

inductively by

$$sd_0(v_0) := id, \quad sd_n(\sigma = (v_0 \cdots v_n)) := \beta_\sigma \circ sd_{n-1} \circ \partial(\sigma). \tag{1.21}$$

In low degrees,

$$sd_1(v_0v_1) = \beta_{(v_0v_1)} \circ sd_0 \circ \partial(v_0v_1) = \beta_{(v_0v_1)} \circ sd_0((v_1) - (v_0)) \tag{1.22}$$

$$= \beta_{(v_0v_1)}((v_1) - (v_0)) = (b_{(v_0v_1)}v_1) - (b_{(v_0v_1)}v_0), \tag{1.23}$$

$$sd_2(v_0v_1v_2) = \beta_{(v_0v_1v_2)} \circ sd_1 \circ \partial(v_0v_1v_2) \tag{1.24}$$

$$= \beta_{(v_0v_1v_2)} \circ sd_1((v_1v_2) - (v_0v_2) + (v_0v_1)) \tag{1.25}$$

$$= \beta_{(v_0v_1v_2)}((b_{(v_1v_2)}v_2) - (b_{(v_1v_2)}v_1) - (b_{(v_0v_2)}v_2) \tag{1.26}$$

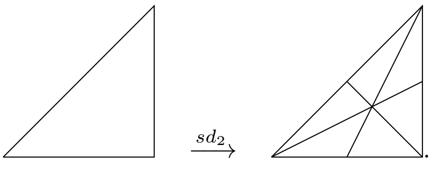
$$+ (b_{(v_0v_2)}v_0) + (b_{(v_0v_1)}v_1) - (b_{(v_0v_1)}v_0)) \tag{1.27}$$

$$= (b_{(v_0v_1v_2)}b_{(v_1v_2)}v_2) - (b_{(v_0v_1v_2)}b_{(v_1v_2)}v_1) \tag{1.28}$$

$$- (b_{(v_0v_1v_2)}b_{(v_0v_2)}v_2) + (b_{(v_0v_1v_2)}b_{(v_0v_2)}v_0) \tag{1.29}$$

$$+ (b_{(v_0v_1v_2)}b_{(v_0v_1)}v_1) - (b_{(v_0v_1v_2)}b_{(v_0v_1)}v_0). \tag{1.30}$$

Thus sd_n is the operation of subdividing the boundary of $\sigma = (v_0 \cdots v_n)$ by barycenters, starting from lower-dimensional simplices. As an example, sd_2 is pictured as



(1.31)

Define another homomorphism

$$D_n : S'_n(\Delta^q) \to S'_{n+1}(\Delta^q) \tag{1.32}$$

inductively by

$$D_0 := 0, \quad D_{n>0}(\sigma) := \beta_\sigma \circ (sd_n - id - D_{n-1} \circ \partial)(\sigma). \tag{1.33}$$

For D_1,

$$D_1(v_0v_1) = \beta_{(v_0v_1)} \circ (sd_1 - id - D_0 \circ \partial)(v_0v_1) \tag{1.34}$$

$$= \beta_{(v_0v_1)}((b_{(v_0v_1)}v_1) - (b_{(v_0v_1)}v_0) - (v_0v_1)) \tag{1.35}$$

$$= (b_{(v_0v_1)}b_{(v_0v_1)}v_1) - (b_{(v_0v_1)}b_{(v_0v_1)}v_0) - (b_{(v_0v_1)}v_0v_1), \tag{1.36}$$

so some vertices coincide. The map D_n is presumably introduced technically in order to prove that sd_n and id are chain homotopic. Formally set $sd_{n<0} = 0$ and $D_{n<0} = 0$.

Lemma 1.2.

- *$sd = (sd_n)$ is a chain map.*
- *$D = (D_n)$ is a chain homotopy from id to sd; that is,*

$$\partial D_n + D_{n-1}\partial = sd_n - id. \tag{1.37}$$

Proof. Since $sd_0\partial = \partial sd_0 = 0$, the claim that sd is a chain map holds for $n = 0$. Assuming it for $i < n$, we get

$$\partial sd_n(\sigma) = \partial\beta_\sigma sd_{n-1}\partial(\sigma) = (1 - \eta_\sigma\epsilon - \beta_\sigma\partial)sd_{n-1}\partial(\sigma) = (1 - \beta_\sigma\partial)sd_{n-1}\partial(\sigma) \tag{1.38}$$

$$= (1 - \beta_\sigma\partial)\partial sd_{n-1}(\sigma) = \partial sd_{n-1}(\sigma), \tag{1.39}$$

where we used

$$\epsilon sd_{n-1}\partial(\sigma^{n>1}) = 0, \quad \epsilon sd_0\partial(v_0v_1) = \epsilon sd_0((v_1) - (v_0)) = 0. \tag{1.40}$$

Next,

$$\partial D_0 + D_{-1}\partial = 0 = sd_0 - id. \tag{1.41}$$

Assume (1.37) for all $i < n$. Then for $n > 0$,

$$(\partial D_n + D_{n-1}\partial)(\sigma) = (\partial\beta_\sigma(sd_n - id - D_{n-1}\partial) + D_{n-1}\partial)(\sigma) \tag{1.42}$$

$$= ((id - \eta_\sigma\epsilon - \beta_\sigma\partial)(sd_n - id - D_{n-1}\partial) + D_{n-1}\partial)(\sigma) \tag{1.43}$$

$$= ((id - \beta_\sigma\partial)(sd_n - id - D_{n-1}\partial) + D_{n-1}\partial)(\sigma) \tag{1.44}$$

$$= (sd_n - id - \beta_\sigma\partial(sd_n - id - D_{n-1}\partial))(\sigma) \tag{1.45}$$

$$= (sd_n - id - \beta_\sigma(sd_{n-1} - id)\partial + \beta_\sigma\partial D_{n-1}\partial)(\sigma) \tag{1.46}$$

$$= (sd_n - id - \beta_\sigma(sd_{n-1} - id)\partial + \beta_\sigma(sd_{n-1} - id - D_{n-2}\partial)\partial)(\sigma) \tag{1.47}$$

$$= (sd_n - id)(\sigma). \tag{1.48}$$

This proves the lemma. □

For a continuous map $f : X \to Y$, the induced map $f_\# : S_*(X) \to S_*(Y)$ is defined by composition:

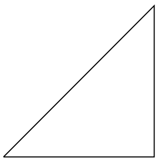
$$(\sigma : \Delta^n \to X) \mapsto (f_\# \circ \sigma : \Delta^n \to X \to Y). \tag{1.49}$$

For a general singular q-simplex $\sigma : \Delta^q \to X$, define

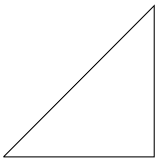
$$sd(\sigma) := \sigma_\# \circ sd_q(\iota) \in S_q(X), \tag{1.50}$$

$$D(\sigma) := \sigma_\# D_q(\iota) \in S_{q+1}(X), \tag{1.51}$$

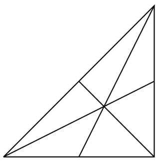
where $\iota = id : \Delta^q \to \Delta^q$. Thus $sd(\sigma)$ is the element of $S_q(X)$ obtained by barycentrically subdividing the domain Δ^q of the singular simplex $\sigma : \Delta^q \to X$:



$$\xrightarrow{\sigma} X \tag{1.52}$$



$$\xrightarrow{sd}$$



$$\xrightarrow{\sigma} X. \tag{1.53}$$

Theorem 1.3.

- *$sd : S(X) \to S(X)$ is a chain map, and $D : S(X) \to S(X)$ is a chain homotopy from id to sd.*
- *The following diagrams commute:*

$$\begin{tikzcd} S(X) \arrow[r, "sd"] \arrow[d, "f_\#"'] & S(X) \arrow[d, "f_\#"'] \\ S(Y) \arrow[r, "sd"] & S(Y) \end{tikzcd} , \quad \begin{tikzcd} S(X) \arrow[r, "D"] \arrow[d, "f_\#"'] & S(X) \arrow[d, "f_\#"'] \\ S(Y) \arrow[r, "D"] & S(Y) \end{tikzcd} . \tag{1.54}$$

The proof is omitted.

- The map $sd : S(X) \to S(X)$ is called the barycentric subdivision operator.

References

[1] Akio Hattori, *Topology*, Iwanami Basic Mathematics Series.