Note: The Bethe Ansatz

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Abstract

For the $M = 2$ sector of the antiferromagnetic XXX model, I derive the Bethe equations for periodic and open boundary conditions. For periodic boundary conditions, I also estimate the number of solutions.

For the XXX model, I derive the Bethe equations mostly following Bethe's original paper [1]. I also referred to [2, 3].

1 The XXX Model

Consider the following Hamiltonian on a circle with N lattice sites, namely the antiferromagnetic Heisenberg chain:

$$H = \frac{1}{2}\sum_{j=1}^{N} \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1} \tag{1}$$

$$= \sum_j (\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + \frac{1}{2}\sigma_j^z \sigma_{j+1}^z) \tag{2}$$

$$= \sum_j (\sigma_{j+1}^- \sigma_j^+ + \sigma_{j-1}^- \sigma_j^+ + 2\frac{1-\sigma_j^z}{2}\frac{1-\sigma_{j+1}^z}{2}) - 2\sum_j \frac{1-\sigma_j^z}{2} + \frac{N}{2}. \tag{3}$$

As a notational convention, set $\boldsymbol{\sigma}_{j+N} = \boldsymbol{\sigma}_j$. The Hamiltonian is $SU(2)$ invariant and translation invariant. Choose a basis that diagonalizes S^z. In the following, as a basis for wave functions, I take $|0\rangle = |\uparrow\uparrow\dots\rangle$ as the "vacuum" and write an M-particle state as

$$|f\rangle_M = \sum_{1\leq x_1 < \cdots < x_M \leq N} f(x_1, \dots, x_M)\sigma_{x_1}^- \dots \sigma_{x_M}^- |0\rangle \tag{4}$$

Note that the eigenfunction f is meaningful only on the domain $1 \leq x_1 < \cdots < x_M \leq N$.

In the sector with particle number M, the term $-2\sum_j \frac{1-\sigma_j^z}{2} = -2M$ is a constant. Hence, in what follows I redefine

$$H = \sum_j (\sigma_{j+1}^- \sigma_j^+ + \sigma_{j-1}^- \sigma_j^+ + 2\frac{1-\sigma_j^z}{2}\frac{1-\sigma_{j+1}^z}{2}) \tag{5}$$

and solve this Hamiltonian.

1.1 Translation Symmetry

(The content of this subsection will not be used later.)

By translation symmetry, the translation operator T defined by

$$T\boldsymbol{\sigma}_j T^{-1} = \boldsymbol{\sigma}_{j+1}, \quad T\left|0\right\rangle = \left|0\right\rangle, \tag{6}$$

and the Hamiltonian H can be diagonalized simultaneously. Since $T^N = 1$, the eigenvalues of T are quantized in $\mathbb{Z}_N$. In the eigenspace with $T = e^{iP}$, $P = \frac{2\pi p}{N}$, $p = 1, \ldots, N-1$, the eigenfunction f satisfies

$$\sum_{1\le x_1<\cdots<x_M\le N} f(x_1,\ldots,x_M)\sigma^-_{x_1+1}\ldots\sigma^-_{x_M+1}\left|0\right\rangle = e^{iP}\sum_{1\le x_1<\cdots<x_M\le N} f(x_1,\ldots,x_M)\sigma^-_{x_1}\ldots\sigma^-_{x_M}\left|0\right\rangle. \tag{7}$$

In the sum on the left-hand side, the contribution with $x_M \le N-1$ is

$$\sum_{1\le x_1<\cdots<x_M\le N-1} f(x_1,\ldots,x_M)\sigma^-_{x_1+1}\ldots\sigma^-_{x_M+1}\left|0\right\rangle. \tag{8}$$

Comparing this with the contribution with $2 \le x_1$ on the right-hand side,

$$e^{iP}\sum_{2\le x_1<\cdots<x_M\le N} f(x_1,\ldots,x_M)\sigma^-_{x_1}\ldots\sigma^-_{x_M}\left|0\right\rangle = e^{iP}\sum_{1\le x_1<\cdots<x_M\le N-1} f(x_1+1,\ldots,x_M+1)\sigma^-_{x_1+1}\ldots\sigma^-_{x_M+1}\left|0\right\rangle \tag{9}$$

we obtain

$$f(x_1,\ldots,x_M) = e^{iP} f(x_1+1,\ldots,x_M+1), \quad (x_M \le N-1) \tag{10}$$

The contribution with $x_M = N$ in the sum on the left-hand side is

$$\sum_{1\le x_1<\cdots<x_{M-1}\le N-1} f(x_1,\ldots,x_{M-1},N)\sigma^-_{x_1+1}\ldots\sigma^-_{x_{M-1}+1}\sigma^-_1\left|0\right\rangle \tag{11}$$

$$= \sum_{1\le x_1<\cdots<x_{M-1}\le N-1} f(x_1,\ldots,x_{M-1},N)\sigma^-_1\sigma^-_{x_1+1}\ldots\sigma^-_{x_{M-1}+1}\left|0\right\rangle \tag{12}$$

and comparing this with the contribution with $x_1 = 1$ on the right-hand side,

$$e^{iP}\sum_{2\le x_2<\cdots<x_M\le N} f(1,x_2,\ldots,x_M)\sigma^-_1\sigma^-_{x_2}\ldots\sigma^-_{x_M}\left|0\right\rangle \tag{13}$$

$$= e^{iP}\sum_{1\le x_2<\cdots<x_M\le N-1} f(1,x_2+1,\ldots,x_M+1)\sigma^-_1\sigma^-_{x_2+1}\ldots\sigma^-_{x_M+1}\left|0\right\rangle \tag{14}$$

$$= e^{iP}\sum_{1\le x_1<\cdots<x_{M-1}\le N-1} f(1,x_1+1,\ldots,x_{M-1}+1)\sigma^-_1\sigma^-_{x_1+1}\ldots\sigma^-_{x_{M-1}+1}\left|0\right\rangle \tag{15}$$

we obtain

$$f(x_1,\ldots,x_{M-1},N) = e^{iP} f(1,x_1+1,\ldots,x_{M-1}+1) \tag{16}$$

1.2 Eigenvalue Equation

Let us examine the structure of the eigenvalue equation

$$H\left|f\right\rangle_M = E\left|f\right\rangle_M \tag{17}$$

1.2.1 $M = 1$

For $M = 1$, the third term drops out, and

$$H|f\rangle_1 = \sum_{j,x}(\delta_{jx}\sigma^-_{j+1} + \delta_{j,x}\sigma^-_{j-1})f(x)|0\rangle \tag{18}$$

$$= \sum_j (f(j)\sigma^-_{j+1} + f(j)\sigma^-_{j-1})|0\rangle \tag{19}$$

$$= \sum_j (f(j-1) + f(j+1))\sigma^-_j |0\rangle \tag{20}$$

therefore the eigenvalue equation is

$$\begin{cases} f(j-1) + f(j+1) = Ef(j) \quad j = 2, \dots, N-1, \\ f(N) + f(2) = Ef(1), \\ f(N-1) + f(1) = Ef(N), \end{cases} \tag{21}$$

The plane wave

$$f_k(x) = e^{ikx} \tag{22}$$

satisfies the first equation, and then $E_k = 2\cos k$. If $k = \frac{2\pi p}{N}$, $p = 0, \dots, N-1$, then e^{ikx} satisfies the second and third equations. Thus we have constructed N linearly independent solutions.

1.2.2 $M = 2$

For the hopping terms, when $x_1 \neq x_2$,

$$\sigma^-_{j\pm1}\sigma^+_j\sigma^-_{x_1}\sigma^-_{x_2}|0\rangle = \delta_{jx_1}\sigma^-_{x_1\pm1}\sigma^-_{x_2}|0\rangle + \delta_{jx_2}\sigma^-_{x_1}\sigma^-_{x_2\pm1}|0\rangle \tag{23}$$

Notice the above relation. When two lowering operators act on the same site, set $\sigma^-_x\sigma^-_x = 0$. For the interaction term, we are using a diagonal basis, and

$$\sum_j \frac{1-\sigma^z_j}{2}\frac{1-\sigma^z_{j+1}}{2}\sigma^-_{x_1}\sigma^-_{x_2}|0\rangle = \delta_{|x_1-x_2|,1}\sigma^-_{x_1}\sigma^-_{x_2}|0\rangle . \tag{24}$$

$\delta_{|x_1-x_2|,1}$ returns 1 only when x_1 and x_2 are adjacent, and returns 0 otherwise. The action of the Hamiltonian on the wave function $|f\rangle_2$ is

$$H|f\rangle_2 = \sum_{1\le x_1<x_2\le N} f(x_1,x_2)(\sigma^-_{x_1+1}\sigma^-_{x_2}|0\rangle + \sigma^-_{x_1}\sigma^-_{x_2+1}|0\rangle + \sigma^-_{x_1-1}\sigma^-_{x_2}|0\rangle + \sigma^-_{x_1}\sigma^-_{x_2-1}|0\rangle + 2\delta_{|x_1-x_2|,1}\sigma^-_{x_1}\sigma^-_{x_2}|0\rangle) \tag{25}$$

Rewrite each term in order. The first term vanishes when $x_2 = x_1 + 1$, so

$$\text{1st.} = \sum_{2\le x_1+1<x_2\le N} f(x_1,x_2)\sigma^-_{x_1+1}\sigma^-_{x_2}|0\rangle = \sum_{2\le x_1<x_2\le N} f(x_1-1,x_2)\sigma^-_{x_1}\sigma^-_{x_2}|0\rangle . \tag{26}$$

For the second term, separate the sum into the case $x_2 = N$ and the remaining cases. Also taking into account that the contribution with $x_1 = 1$ vanishes when $x_2 = N$, we have

$$\text{2nd.} = \sum_{1\le x_1<x_2\le N-1} f(x_1,x_2)\sigma^-_{x_1}\sigma^-_{x_2+1}|0\rangle + \sum_{2\le x_1\le N-1} f(x_1,N)\sigma^-_{x_1}\sigma^-_1|0\rangle \tag{27}$$

$$= \sum_{1\le x_1<x_2-1\le N-1} f(x_1,x_2-1)\sigma^-_{x_1}\sigma^-_{x_2}|0\rangle + \sum_{2\le x_2\le N-1} f(x_2,N)\sigma^-_1\sigma^-_{x_2}|0\rangle . \tag{28}$$

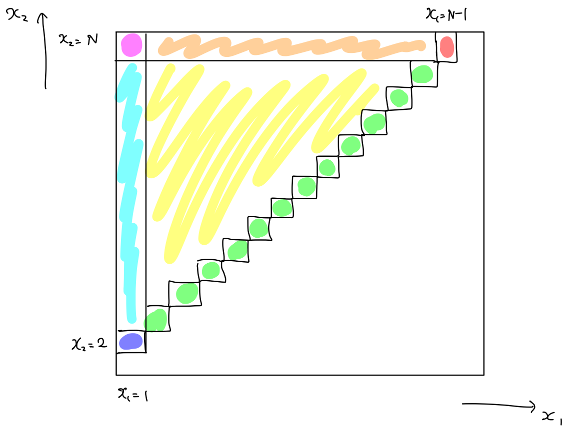

Figure 1

For the third term, separate the sum into the case $x_1 = 1$ and the remaining cases. Moreover, when $x_1 = 1$, the contribution with $x_2 = N$ vanishes, and hence

$$\text{3rd.} = \sum_{2\le x_1<x_2\le N} f(x_1,x_2)\sigma^-_{x_1-1}\sigma^-_{x_2}|0\rangle + \sum_{2\le x_2\le N-1} f(1,x_2)\sigma^-_N\sigma^-_{x_2}|0\rangle \tag{29}$$

$$= \sum_{1\le x_1<x_2-1\le N-1} f(x_1+1,x_2)\sigma^-_{x_1}\sigma^-_{x_2}|0\rangle + \sum_{2\le x_1\le N-1} f(1,x_1)\sigma^-_{x_1}\sigma^-_N|0\rangle\,. \tag{30}$$

The fourth term vanishes when $x_2 = x_1 + 1$, so

$$\text{4th.} = \sum_{2\le x_1+1<x_2\le N} f(x_1,x_2)\sigma^-_{x_1}\sigma^-_{x_2-1}|0\rangle = \sum_{1\le x_1<x_2\le N-1} f(x_1,x_2+1)\sigma^-_{x_1}\sigma^-_{x_2}|0\rangle\,. \tag{31}$$

The fifth term contributes only when the two points are adjacent on the circle, and

$$\text{5th.} = \sum_{1\le x\le N-1} f(x,x+1)2\sigma^-_x\sigma^-_{x+1}|0\rangle + f(1,N)2\sigma^-_1\sigma^-_N|0\rangle\,. \tag{32}$$

The case distinction in the sums is illustrated in Figure 1. Let us look at the eigenvalue equation in each region.

- For $2 \le x_1$, $x_1 + 1 \le x_2 \le N - 1$ (yellow),

$$(A): \quad f(x_1-1,x_2) + f(x_1,x_2-1) + f(x_1+1,x_2) + f(x_1,x_2+1) = Ef(x_1,x_2). \tag{33}$$

- For $x_1 = 1$, $3 \le x_2 \le N - 1$ (cyan),

$$(B): \quad f(x_2,N) + f(1,x_2-1) + f(2,x_2) + f(1,x_2+1) = Ef(1,x_2). \tag{34}$$

- For $2 \le x_1 \le N - 2$, $x_2 = N$ (orange),

$$(C): \quad f(x_1-1,N) + f(x_1,N-1) + f(x_1+1,N) + f(1,x_1) = Ef(x_1,N). \tag{35}$$

- For $2 \le x_1 = x$, $x_2 = x + 1$, $2 \le x \le N - 2$ (green),

$$(D): \quad f(x-1,x+1) + f(x,x+2) + 2f(x,x+1) = Ef(x,x+1). \tag{36}$$

- For $x_1 = 1, x_2 = 2$ (blue),

$$(E): \quad f(2,N) + f(1,3) + 2f(1,2) = Ef(1,2). \tag{37}$$

- For $x_1 = N-1, x_2 = N$ (red),

$$(F): \quad f(N-2,N) + f(1,N-1) + 2f(N-1,N) = Ef(N-1,N). \tag{38}$$

- For $x_1 = 1, x_2 = N$ (pink),

$$(G): \quad f(1,N-1) + f(2,N) + 2f(1,N) = Ef(1,N). \tag{39}$$

Now, looking at the equations, it is convenient to extend the domain $1 \le x_1 < x_2 \le N$ of $f(x_1,x_2)$ so that it includes $0, N+1, N+2$, and to introduce boundary conditions for the extra degrees of freedom of $f(x_1,x_2)$. Introduce the following $2N-1$ new variables:

$$f(0,x_2), \quad x_2 = 1,\ldots,N-1, \tag{40}$$
$$f(x_1,N+1), \quad x_1 = 2,\ldots,N, \tag{41}$$
$$f(N,N+2), \tag{42}$$

Furthermore, impose the following "boundary conditions":

$$\begin{cases} f(0,x_2) = f(x_2,N), & x_2 = 1,\ldots,N-1, \\ f(x_1,N+1) = f(1,x_1), & x_1 = 2,\ldots,N, \\ f(N,N+2) = f(2,N). \end{cases} \tag{43}$$

These are natural relations for a function on the circle. Under these boundary conditions, the equations (A,B,C) can be combined as

$$(\text{Eq.1}): \quad f(x_1-1,x_2) + f(x_1,x_2-1) + f(x_1+1,x_2) + f(x_1,x_2+1) = Ef(x_1,x_2), \quad (|x_1-x_2| \ge 2). \tag{44}$$

and similarly (D,E,F,G) can be rewritten as

$$(\text{Eq.2}): \quad f(x-1,x+1) + f(x,x+2) + 2f(x,x+1) = Ef(x,x+1), \quad (x = 1,\ldots,N), \tag{45}$$

For the extension of variables, it is equivalently possible to introduce infinitely many variables

$$f(x_1,x_2), \quad x_1,x_2 \in \mathbb{Z}, \quad x_1 < x_2 < x_1+N, \tag{46}$$

and impose the boundary condition

$$(\text{B.C.}): \quad f(x_1,x_2) = f(x_2,x_1+N) = f(x_2-N,x_1) \tag{47}$$

instead. Thus the problem is to solve the eigenvalue equations (Eq.1, Eq.2), under the boundary condition (B.C.), for variables $f(x_1,x_2)$ satisfying $x_1 < x_2 < x_1+N$.

Following Bethe [1], we find eigenfunctions. Since (Eq.1) has no interaction term, one expects a "two-particle plane-wave state" to be a solution. For example, for arbitrary complex numbers k_1, k_2,

$$e^{ik_1x_1}e^{ik_2x_2}, \quad e^{ik_2x_1}e^{ik_1x_2} \tag{48}$$

are both solutions of (Eq.1) with $E = 2\cos k_1 + 2\cos k_2$. However,

$$f(x_2,x_1+N) = e^{ik_2N}e^{ik_1x_2}e^{ik_2x_1} \tag{49}$$

so the boundary condition cannot be satisfied for general (k_1,k_2). Therefore, for $k_1 \ne k_2$, assume a solution of the form

$$g_{\boldsymbol{k}}(x_1,x_2) = c_1e^{i(k_1x_1+k_2x_2)} + c_2e^{i(k_2x_1+k_1x_2)}, \quad x_1 < x_2 < x_1+N, \tag{50}$$

and determine $\boldsymbol{k} = (k_1, k_2)$ and the relative ratio of c_1, c_2 from (Eq.2) and (B.C.). Substituting $g_{\boldsymbol{k}}(x_1, x_2)$ into (Eq.2), we get

$$g_{\boldsymbol{k}}(x-1, x+1) + g_{\boldsymbol{k}}(x, x+2) + 2g_{\boldsymbol{k}}(x, x+1) = E_{\boldsymbol{k}} g_{\boldsymbol{k}}(x, x+1) \tag{51}$$

We reinterpret this equation as follows. First introduce the function

$$h_{\boldsymbol{k}}(x)(= \text{“}g_{\boldsymbol{k}}(x,x)\text{”}) := c_1 e^{i(k_1 x + k_2 x)} + c_2 e^{i(k_2 x + k_1 x)} = e^{i(k_1+k_2)x}(c_1 + c_2) \tag{52}$$

Note that $g_{\boldsymbol{k}}(x_1, x_2)$ satisfies (Eq.1) for arbitrary (x_1, x_2) outside the original domain. Then the relation

$$g_{\boldsymbol{k}}(x-1, x+1) + h_{\boldsymbol{k}}(x) + h_{\boldsymbol{k}}(x+1) + g_{\boldsymbol{k}}(x, x+2) = E_{\boldsymbol{k}} g_{\boldsymbol{k}}(x, x+1) \tag{53}$$

holds. Using this, equation (51) is equivalent to

$$h_{\boldsymbol{k}}(x) + h_{\boldsymbol{k}}(x+1) - 2g_{\boldsymbol{k}}(x, x+1) = 0, \tag{54}$$

namely

$$(1 + e^{i(k_1+k_2)})c_1 + (1 + e^{i(k_1+k_2)})c_2 - 2c_1 e^{ik_2} - 2c_2 e^{ik_1} = 0 \tag{55}$$

Solving this, we obtain the relative ratio

$$\frac{c_1}{c_2} = -\frac{1 + e^{i(k_1+k_2)} - 2e^{ik_1}}{1 + e^{i(k_1+k_2)} - 2e^{ik_2}} \tag{56}$$

Here I assumed $c_2 \neq 0$.

The boundary condition (B.C.), $g_{\boldsymbol{k}}(x_2, x_1 + N) = g_{\boldsymbol{k}}(x_1, x_2)$, gives

$$(c_1 - c_2 e^{ik_1 N}) e^{i(k_1 x_1 + k_2 x_2)} + (c_2 - c_1 e^{ik_2 N}) e^{i(k_1 x_2 + k_2 x_1)} = 0, \quad x_1 < x_2 < x_1 + N. \tag{57}$$

Taking into account the assumption $k_1 \neq k_2$, the condition for this to hold for arbitrary $x_1 < x_2 < x_1 + N$ is

$$\begin{cases} c_1 = c_2 e^{ik_1 N}, \\ c_2 = c_1 e^{ik_2 N}, \end{cases} \tag{58}$$

Thus we obtain two equations, the Bethe equations, that determine the unknown variables (k_1, k_2):

$$e^{ik_1 N} = -\frac{1 + e^{i(k_1+k_2)} - 2e^{ik_1}}{1 + e^{i(k_1+k_2)} - 2e^{ik_2}}, \tag{59}$$

$$e^{ik_2 N} = -\frac{1 + e^{i(k_1+k_2)} - 2e^{ik_2}}{1 + e^{i(k_1+k_2)} - 2e^{ik_1}}, \tag{60}$$

Here one should keep in mind that $k_1 \neq k_2$.

Because of $\varphi_{\boldsymbol{k}}$, k_1, k_2 do not take the usual wavenumber form. However, noting that

$$e^{i(k_1+k_2)N} = 1 \tag{61}$$

we find for the sum that

$$k_1 + k_2 = \frac{2\pi(\lambda_1 + \lambda_2)}{N} \tag{62}$$

holds. The quantity $k_1 + k_2$ is nothing but the eigenvalue of the translation operator.

1.2.3 The Case $k_1 = k_2$

When $k_1 = k_2 = k$, if we take the ansatz

$$g_{k,k}(x_1, x_2) = e^{ik(x_1+x_2)} \tag{63}$$

then (Eq.1) and (Eq.2) give, respectively,

$$E = 4\cos k, \quad 2\cos k + 2 = E, \tag{64}$$

and hence we obtain $k = 0$, namely the constant solution $f(x_1, x_2) = \text{const.}$. Note that the $k = 0$ solution is special to the $SU(2)$-symmetric case. If, in the XXZ model, the coefficient of $\sigma_j^z \sigma_{j+1}^z$ is multiplied by Δ, then the equation becomes

$$\cos k = \Delta \tag{65}$$

and the behavior is as follows:

- For $\Delta < -1$, there is no solution.
- For $\Delta = -1$, there is the single solution $k = \pi$.
- For $-1 < \Delta < 1$, there are two real solutions $k = \pm\cos^{-1}(\Delta)$.
- For $\Delta = 1$, there is the single solution $k = 0$.
- For $\Delta > 1$, there are two purely imaginary solutions $k = \pm\cosh^{-1}(\Delta)$.

On the other hand, the boundary condition (B.C.) gives

$$e^{ikN} = 1 \tag{66}$$

and therefore the purely imaginary solutions are not realized. For $-1 \leq \Delta \leq 1$, a solution exists only for special values of Δ determined by N.

For the XXX model ($\Delta = 1$),

- there is one constant solution with $k = 0$, independently of N.

1.2.4 Symmetries of the Bethe Equations

Let us record several properties of the Bethe equations.

$$e^{i\varphi_{\boldsymbol{k}}} = \frac{c_1}{c_2} = -\frac{1 + e^{i(k_1+k_2)} - 2e^{ik_1}}{1 + e^{i(k_1+k_2)} - 2e^{ik_2}}, \quad k_1 \neq k_2, \tag{67}$$

$$e^{i\varphi_{k_2,k_1}} = e^{-i\varphi_{k_1,k_2}}, \tag{68}$$

$$e^{i\varphi_{-k_1,-k_2}} = e^{i\varphi_{k_2,k_1}} = e^{-i\varphi_{k_1,k_2}}, \tag{69}$$

$$[e^{i\varphi_{k_1,k_2}}]^* = e^{i\varphi_{k_2^*,k_1^*}} = e^{-i\varphi_{k_1^*,k_2^*}}, \tag{70}$$

$$e^{i\varphi_{k,k}} = -1, \tag{71}$$

$$e^{i\varphi_{0,k_2}} = e^{i\varphi_{k_1,0}} = 1, \tag{72}$$

Note these identities. Also note that the function $e^{i\varphi_{k,k}}$ as a function of k is discontinuous at $k = 0$.

$$f_1(k_1, k_2) = e^{ik_1 N - \varphi_{k_1,k_2}}, \quad f_2(k_1, k_2) = e^{ik_2 N + \varphi_{k_1,k_2}} \tag{73}$$

If we set the above, the Bethe equations are

$$f_1(k_1,k_2) = f_2(k_1,k_2) = 1. \tag{74}$$

Since $f_1(k_2,k_1) = f_2(k_1,k_2)$ and $f_2(k_2,k_1) = f_1(k_1,k_2)$,

- the equations are symmetric under the interchange $(k_1,k_2) \to (k_2,k_1)$. In other words, (k_1,k_2) is a solution if and only if (k_2,k_1) is a solution. After all, the Bethe ansatz solution $g_{\boldsymbol{k}}$ was specified by a pair of two distinct wavenumbers (k_1,k_2).

Since $f_1(-k_1,-k_2) = f_1(k_1,k_2)^{-1}$ and $f_2(-k_1,-k_2) = f_2(k_1,k_2)^{-1}$,

- the equations are symmetric under $(k_1,k_2) \to (-k_1,-k_2)$. In other words, (k_1,k_2) is a solution if and only if $(-k_1,-k_2)$ is a solution. This presumably comes from inversion symmetry of the system.

Since $f_1(k_1,k_2)^* = f_1(k_1^*,k_2^*)^{-1}$ and $f_2(k_1,k_2)^* = f_2(k_1^*,k_2^*)^{-1}$,

- the equations are symmetric under $(k_1,k_2) \to (k_1^*,k_2^*)$. In other words, (k_1,k_2) is a solution if and only if (k_1^*,k_2^*) is a solution.
- Note that the solutions have the character of "PT-symmetry breaking". That is, if one extends the XXX model to a parameter-dependent model while preserving this symmetry, complex solutions are generated by the collision of two real solutions.

Since $f_1(k,k) = f_2(k,k) = -1$,

- $k_1 = k_2 = \frac{2\pi(n-1/2)}{N}$, $n = 1,\dots,N$, are solutions.

Since $f_1(0,k_2) = 1$ and $f_2(0,k_2) = e^{ik_2N}$,

- $(k_1,k_2) = (0, \frac{2\pi n}{N})$, $n = 0,\dots,N-1$, are solutions.

1.2.5 Real Solutions

Following Bethe [1], let us count the number of independent solutions when k_1,k_2 are real.

$$e^{i\varphi_{\boldsymbol{k}}} = -\frac{e^{-i(k_1+k_2)/2} + e^{i(k_1+k_2)/2} - 2e^{i(k_1-k_2)/2}}{e^{-i(k_1+k_2)/2} + e^{i(k_1+k_2)/2} - 2e^{-i(k_1-k_2)/2}} = -\frac{\cos\frac{k_1+k_2}{2} - e^{-i(k_2-k_1)/2}}{\cos\frac{k_1+k_2}{2} - e^{i(k_2-k_1)/2}} \tag{75}$$

As is clear from this rewriting, if k_1,k_2 are real, then $\varphi_{\boldsymbol{k}}$ is also real. The Bethe equations can be rewritten as

$$\begin{cases} Nk_1 - \varphi_{\boldsymbol{k}} = 0 \mod 2\pi, \\ Nk_2 + \varphi_{\boldsymbol{k}} = 0 \mod 2\pi, \end{cases} \tag{76}$$

Let us examine the properties of $\varphi_{\boldsymbol{k}}$:

$$\begin{aligned} \varphi_{\boldsymbol{k}} &= \pi + 2\arg\left(\cos\frac{k_1+k_2}{2} - e^{-i(k_2-k_1)/2}\right) && (77)\\ &= \pi + 2\arg\left(-2\sin\frac{k_1}{2}\sin\frac{k_2}{2} + i\sin\frac{k_2-k_1}{2}\right) && (78)\\ &= \pi + 2\arg\left(2\sin\frac{k_1}{2}\sin\frac{k_2}{2} - i\sin\frac{k_2-k_1}{2}\right) \mod 2\pi, && (79) \end{aligned}$$

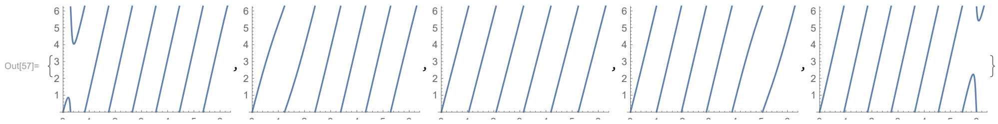

Figure 2: The function $Nk_1 - \varphi_{\boldsymbol{k}} \bmod 2\pi$, with $N = 7$. From left to right, $k_2 = \pi/10, \pi/2, \pi, 3\pi/2, 2\pi - \pi/10$.

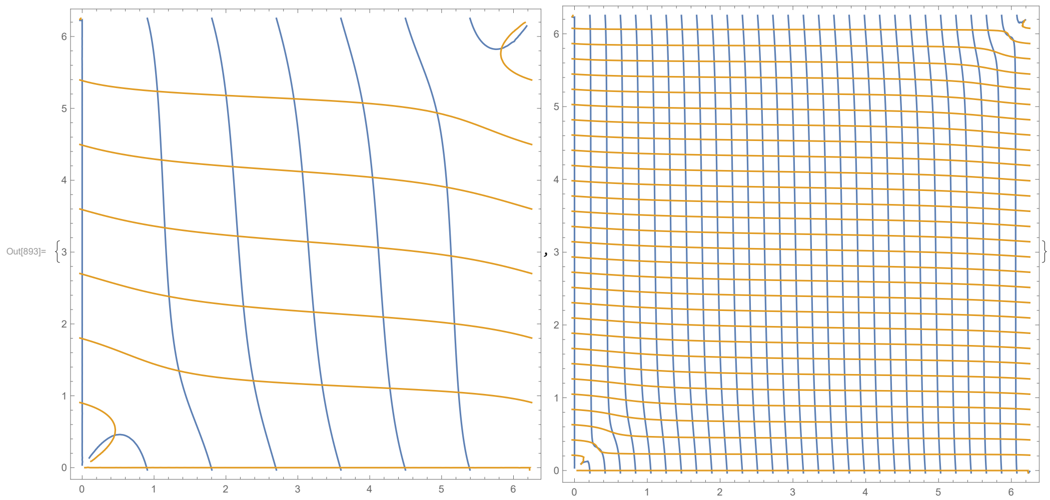

Figure 3: Solution curves for $Nk_1 - \varphi_{\boldsymbol{k}} \equiv 0$ (blue) and $Nk_2 + \varphi_{\boldsymbol{k}} \equiv 0$ (yellow). The horizontal axis is k_1, and the vertical axis is k_2. The precision is MachinePrecision. Solutions (k_1, k_2) are obtained as intersections of the two solution curves. The two panels are for $N = 7, 30$, respectively.

Thus, when k_1 is varied from 0 through k_2 to 2π, $\varphi_{\boldsymbol{k}}$ increases monotonically as $\varphi_{0,k_2} = 0 \to \varphi_{k_2,k_2} = \pi \to \varphi_{2\pi,k_2} = 2\pi$. Therefore, for any k_2, the function $Nk_1 - \varphi_{\boldsymbol{k}}$ has winding number $N-1$ as k_1 winds once from 0 to 2π. Hence the equation $Nk_1 - \varphi_{\boldsymbol{k}} \equiv 0$ for k_1 has at least $N-1$ solutions, independently of k_2. Note that, in general, there may also be an even number of additional solutions that are unstable under deformations of k_2. Figure 2 shows plots of the function $Nk_1 - \varphi_{\boldsymbol{k}}$ for several choices of k_2 in the case $N = 7$. From Figure 2, one sees that multiple solutions in k_1 tend to appear when k_2 is close to $0 = 2\pi$.

Do the unstable solutions not protected by the winding number contribute to solutions of the Bethe equations? To check this, Figure 3 shows the zeros of $Nk_1 - \varphi_{\boldsymbol{k}}$ and $Nk_2 + \varphi_{\boldsymbol{k}}$. From the figure, one sees that the smallest point $(k_1 = 0, k_2^{\min})$ with $k_2 > 0$ at $k_1 = 0$ arises from an unstable zero not fixed by the winding number. Moreover, considering the stability of the existence of this solution $(0, k_2^{\min})$ with respect to N, since $(k_1, k_2) = (\frac{\pi}{N}, \frac{\pi}{N})$ is a solution, the solution $(0, k_2^{\min})$ is expected to remain for arbitrary N.

Assume that intersections such as those near $k_1 = k_2 = \frac{2\pi(2-1/2)}{N}$ in the right panel of Figure 3 are not true solutions because of numerical precision (this will later turn out to be wrong). Then the number of solutions satisfying $0 \le k_1 < k_2 < 2\pi$ is

$$(\sum_{n=1}^{N-2} n) + 1 = \frac{(N-1)(N-2)}{2} + 1 \tag{80}$$

This differs from Bethe's estimate [1], $\frac{(N-1)(N-2)}{2}$, only by the contribution from the unstable zero. Below I also record Bethe's argument.

1.2.6 Bethe's Estimate of the Number of Real Solutions

Let us follow the argument in [1].

The solutions are

$$k_1 = \frac{2\pi\lambda_1}{N} + \frac{\varphi_{\boldsymbol{k}}}{N}, \tag{81}$$

$$k_2 = \frac{2\pi\lambda_2}{N} - \frac{\varphi_{\boldsymbol{k}}}{N}, \tag{82}$$

$$\lambda_1, \lambda_2 = 0, \ldots, N-1. \tag{83}$$

$\varphi_{\boldsymbol{k}}$ may be taken to satisfy

$$-\pi \le \varphi_{\boldsymbol{k}} \le \pi \tag{84}$$

$$k_2 - k_1 = \frac{2\pi(\lambda_2 - \lambda_1)}{N} - \frac{2\varphi_{\boldsymbol{k}}}{N} \tag{85}$$

Therefore, for $\lambda_1 = 0, \ldots, \lambda_2 - 2$, the inequality $0 \le k_1 < k_2$ is satisfied. However, when $\lambda_1 = \lambda_2 - 1$, $k_1 < k_2$ may fail. Indeed, if one formally solves the Bethe equations $e^{i(k_1 N - \varphi_{\boldsymbol{k}})} = e^{i(k_2 N + \varphi_{\boldsymbol{k}})} = 1$ for the case $k_1 = k_2$, one obtains the solution $\varphi_{\boldsymbol{k}} = \pi$, $\lambda_1 = \lambda_2 - 1$. If the Bethe equation $e^{ik_1 N} = e^{i\varphi_{k_1,k_2}}$ is regarded as an equation determining k_1 for fixed k_2, the monotonicity of $\varphi_{\boldsymbol{k}}$ on $0 \le k_1 \le k_2$ shows that, for a fixed branch λ_1, there is at most one solution. Hence $\lambda_1 = \lambda_2 - 1$ is inappropriate as a solution. Thus the number of independent ansatz solutions $g_{\boldsymbol{k}}(x_1, x_2)$ is

$$\sum_{\lambda_2=2}^{N-1} \sum_{\lambda_1=0}^{\lambda_2-2} = \sum_{\lambda_2=2}^{N-1} (\lambda_2 - 1) = \frac{(N-1)(N-2)}{2}. \tag{86}$$

This argument misses the fact that, due to the singular behavior of $\varphi_{\boldsymbol{k}}$ at $k_1 = 0$, a solution with $\lambda_1 = 0$ may exist when $\lambda_2 = 1$. Indeed, if $k_1 = 0$ and $k_1 < k_2$, then the Bethe equations give $\lambda_1 = 0$ and

$$k_2 = \frac{2\pi\lambda_2}{N}. \tag{87}$$

This has solutions for $\lambda_2 = 1, \ldots, N-1$.

1.2.7 Complex-Conjugate Solutions

The number of real solutions is $\frac{(N-1)(N-2)}{2} + 1$, and the number of constant solutions is one when N is even and zero when N is odd. Since the dimension of the $M = 2$ Hilbert space is $\frac{N(N-1)}{2}$, the number of solutions is short by $N - 2$ when N is even and by $N - 3$ when N is odd.

Since the total momentum $k_1 + k_2$ is real, we may set

$$\begin{cases} k_1 = u + k, \\ k_2 = u - k, \end{cases} \tag{88}$$

Following [1], first look for solutions in which k_1, k_2 are complex conjugates. That is, set $k = iv$ with v real. It is enough to study $v > 0$.

$$e^{i\varphi_{\boldsymbol{k}}} = -\frac{\cos\frac{k_1+k_2}{2} - e^{-i(k_2-k_1)/2}}{\cos\frac{k_1+k_2}{2} - e^{i(k_2-k_1)/2}} = -\frac{\cos u - e^{-v}}{\cos u - e^{v}} \tag{89}$$

Then the Bethe equations become

$$\left\{ \begin{array}{l} e^{iuN-vN} = e^{i\varphi_{\boldsymbol{k}}} = -\frac{\cos u - e^{-v}}{\cos u - e^{v}}, \\ e^{iuN+vN} = e^{-i\varphi_{\boldsymbol{k}}} = -\frac{\cos u - e^{v}}{\cos u - e^{-v}}. \end{array} \right. \tag{90}$$

Thus, introducing the function

$$F(u,v) := \frac{e^{v} - \cos u}{e^{-v} - \cos u} \tag{91}$$

we have

$$e^{iuN} = -\mathrm{Sign}F(u,v), \tag{92}$$

$$Nv = \log |F(u,v)|. \tag{93}$$

The second equation (93) can be rewritten as

$$\frac{e^{v} - \cos u}{e^{-v} - \cos u} = e^{2v}\frac{e^{-v} - \cos u e^{-2v}}{e^{-v} - \cos u} \tag{94}$$

From this, when $-\cos u \geq 0$ and $N > 2$, there is no solution other than $v = 0$. Therefore it is enough to consider the region $\cos u > 0$. Also, when $\cos u = 1$, $\log |F(u,v)| = v$, so for $N > 1$ there is no solution. Thus it remains to consider only $0 < \cos u < 1$. In this case, the function $\log |F(u,v)|$ diverges to $+\infty$ at $v = -\log \cos u$ and is asymptotic to $v+$const.. Moreover,

$$\partial_v \log F(u,v) = \frac{e^{v}(e^{-v} - \cos u) + (1 - e^{-v}\cos u)}{(e^{v} - \cos u)(e^{-v} - \cos u)} > 0, \quad (0 < v < -\log \cos u), \tag{95}$$

$$\partial_v^2 \log F(u,v) = \frac{e^{v}(e^{2v} - 1)\cos u \sin^2 u}{(e^{v} - \cos u)^2(e^{v}\cos u - 1)^2} > 0, \quad (0 < v < -\log \cos u), \tag{96}$$

so it is convex and monotonically increasing for $0 < v < -\log \cos u$. The slope at $v = 0$ is $\frac{2}{1-\cos u}$, and hence

- if $N < \frac{2}{1-\cos u}$, there is one intersection for $v \gtrsim -\log \cos u$.
- if $N > \frac{2}{1-\cos u}$, there is one intersection each for $v \lesssim -\log \cos u$ and $v \gtrsim -\log \cos u$.

Next determine u from the first equation (92). Since $e^{2iuN} = 1$, a necessary condition is

$$u = \frac{\pi n}{N}, \quad n \in \mathbb{Z} \tag{97}$$

At this value, $e^{iuN} = (-1)^n$, so if (u,v) is a solution then $(-u,v)$ is also a solution. Since $e^{v} - \cos u > 0$, the sign change on the right-hand side of (92) occurs at $v = -\log \cos u$, which is the divergence point of the right-hand side of (93). From the condition $0 < \cos u < 1$,

- when N is odd, the candidates are $u_n = \pm\frac{\pi n}{N}$, $n = 1, \ldots, \frac{N-1}{2}$;
- when N is even, the candidates are $u_n = \pm\frac{\pi n}{N}$, $n = 1, \ldots, \frac{N-2}{2}$.

Figure 4 plots the functions Nv and $\log |F(u,v)|$ for $u = u_n$, $n = 1, \ldots, \mathrm{Floor}[\frac{N-1}{2}]$, for $N = 7, 20$. Equation (92) is

$$(-1)^n = \left\{ \begin{array}{ll} -1 & (0 < v < -\log \cos u_n), \\ 1 & (-\log \cos u_n < v), \end{array} \right. \tag{98}$$

and hence

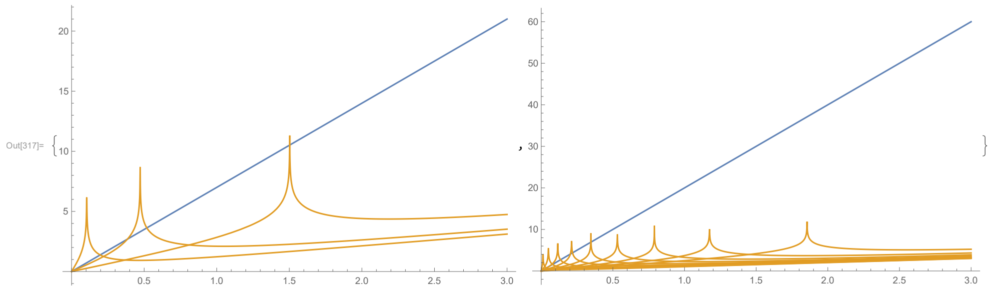

Figure 4: Plots of Nv (blue) and $\log|F(u,v)|$ (yellow) for $u = u_n$, $n = 1, \dots, \mathrm{Floor}(\frac{N-1}{2})$, for $N = 7$ (left) and $N = 20$ (right).

- for odd n, the condition for a solution is $0 < v < -\log\cos u_n$;
- for even n, the condition for a solution is $-\log\cos u_n < v$.

Taking this into account,

- if n is odd and $N > \frac{2}{1-\cos u_n}$ is satisfied, there is one solution for $v \lesssim -\log\cos u_n$.
- if n is odd and $N < \frac{2}{1-\cos u_n}$ is satisfied, there is no solution.
- if n is even, there is one solution for $v \gtrsim -\log\cos u_n$.

The condition $N < \frac{2}{1-\cos u_n}$ is equivalent to

$$n < \frac{2N}{\pi}\sin^{-1}(\frac{1}{\sqrt{N}}) \tag{99}$$

The number of odd integers $n = 2m-1$, $m = 1, 2, \dots$, satisfying this condition is M_N, where M_N is the largest integer not exceeding $\frac{N}{\pi}\sin^{-1}(\frac{1}{\sqrt{N}}) + \frac{1}{2}$. For $N \gg 1$,

$$M_N \sim \frac{\sqrt{N}}{\pi} \tag{100}$$

Computing M_N, one obtains

$$M_N = \begin{cases} 1 & (N \leq 21) \\ 2 & (22 \leq N \leq 61) \\ 3 & (62 \leq N \leq 120) \\ 4 & (121 \leq N \leq 199) \\ 5 & (200 \leq N \leq 298) \\ 6 & (299 \leq N \leq 416) \\ \vdots & \end{cases} \tag{101}$$

and so on. As examples, Figure 5 shows the functions $\log|F(u,v)|$ for $u = u_1, u_3$ at $N = 10, 21, 40$.

Therefore, the number of complex-conjugate solutions is, with a factor of two from the contributions of $u = u_n$ and $-u_n$,

- when N is odd, $N - 1 - 2M_N$.
- when N is even, $N - 2 - 2M_N$.

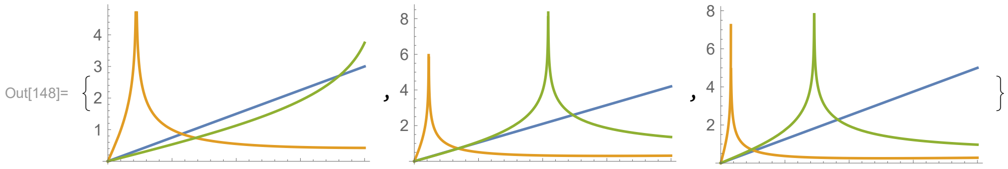

Figure 5: The functions Nv (blue), $\log|F(u_1, v)|$ (yellow), and $\log|F(u_3, v)|$ (green) for $N = 10, 21, 40$.

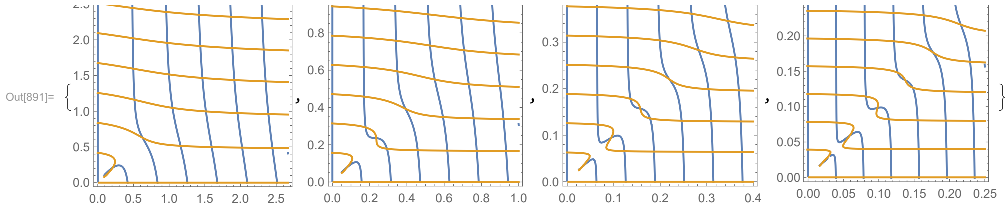

Figure 6: Solution curves for $Nk_1 - \varphi_{\boldsymbol{k}} \equiv 0$ (blue) and $Nk_2 + \varphi_{\boldsymbol{k}} \equiv 0$ (yellow). The horizontal axis is k_1, and the vertical axis is k_2. Only the neighborhood of $(k_1, k_2) = (0, 0)$ is shown. From left to right, $N = 15, 40, 100, 160$.

1.2.8 Real Solutions with Close Wavenumbers

Compared with the Hilbert-space dimension $N(N-1)/2$, the number of solutions is short by approximately $2M_N \sim \frac{2\sqrt{N}}{\pi}$ for large N. Looking at [3, 4] and related references, it seems that additional real solutions exist. In Section 1.2.5, I ignored intersections with close wavenumbers near $(k_1, k_2) = \frac{2\pi(2-1/2)}{N}, \frac{2\pi(N-1-1/2)}{N}$ for $N = 20$, but these are in fact solutions. See Figure 6.

By plotting neighborhoods of $k_1 = k_2 = \frac{2\pi(n-1/2)}{N}$, $n = 1, 2, 3, \ldots$, and checking by eye when such solutions appear, I find:

- For $N = 22$, they appear near $k_1 = k_2 = \pm\frac{2\pi(2-1/2)}{N}$.
- For $N = 62$, they appear near $k_1 = k_2 = \pm\frac{2\pi(3-1/2)}{N}$.
- For $N = 121$, they appear near $k_1 = k_2 = \pm\frac{2\pi(4-1/2)}{N}$.
- For $N = 200$, they appear near $k_1 = k_2 = \pm\frac{2\pi(5-1/2)}{N}$.
- For $N = 299$, they appear near $k_1 = k_2 = \pm\frac{2\pi(6-1/2)}{N}$.
- For $N = 417$, they appear near $k_1 = k_2 = \pm\frac{2\pi(7-1/2)}{N}$.

The values of N at which new solutions appear agree exactly with $M_N - 1$ in the previous subsection. Therefore, the number of real solutions with close wavenumbers is

$$2(M_N - 1) \tag{102}$$

I leave a rigorous computation of the number of close-wavenumber solutions for later.

1.2.9 Counting Bethe-Ansatz Solutions

Summarizing the above,

- there is one constant solution.
- the number of real solutions with separated wavenumbers is $\frac{(N-1)(N-2)}{2}+1$.
- the number of complex-conjugate solutions is
  - for odd N, $N-1-2M_N$.
  - for even N, $N-2-2M_N$.
- the number of real solutions with close wavenumbers is $2(M_N-1)$.

Thus the number of Bethe-ansatz solutions is

- for odd N, $\frac{N(N-1)}{2}$.
- for even N, $\frac{N(N-1)}{2}-1$.

Therefore, when N is even, one solution is missing.

1.2.10 The Case $N=3$

For $N=3$, there are two real solutions, $(k_1,k_2)=(0,2\pi/3),(0,4\pi/3)$, and one constant solution. Let us solve the eigenvalue equation directly and compare with the Bethe-ansatz solutions. There are variables $\boldsymbol{f}=(f(1,2),f(1,3),f(2,3))^T$, and the eigenvalue equation is

$$\begin{cases} f(2,3)+f(1,3)+2f(1,2)=Ef(1,2), \\ f(1,2)+f(2,3)+2f(1,3)=Ef(1,3), \\ f(1,3)+f(1,2)+2f(2,3)=Ef(2,3). \end{cases} \tag{103}$$

that is,

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \boldsymbol{f}=E\boldsymbol{f}. \tag{104}$$

Solving this, we obtain the following three solutions:

$$\boldsymbol{f}=(1,1,1)^T, \quad E=4, \tag{105}$$
$$\boldsymbol{f}=(1,0,-1)^T, \quad E=1, \tag{106}$$
$$\boldsymbol{f}=(1,-1,0)^T, \quad E=1. \tag{107}$$

These correspond to the constant solution and the Bethe-ansatz solutions. [1]

[1]

$$g_{0,\frac{2\pi}{3}}(x_1,x_2)=e^{\frac{2\pi i}{3}x_1}+e^{\frac{2\pi i}{3}x_2}=(-1,e^{\pi i/3},e^{-\pi i/3}), \tag{108}$$
$$g_{0,\frac{4\pi}{3}}(x_1,x_2)=e^{\frac{2\pi i}{3}x_1}+e^{\frac{2\pi i}{3}x_2}=(-1,e^{-\pi i/3},e^{\pi i/3}). \tag{109}$$

Linear combinations of these.

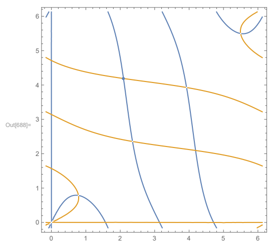

Figure 7: Real solutions of the Bethe equations for $N = 4$. The point $(\frac{2\pi}{3}, \frac{4\pi}{3})$ is also plotted.

1.2.11 The Case $N = 4$

On the other hand, for $N = 4$, the above argument gives four real solutions, $(k_1, k_2) = (0, 2\pi/4), (0, 4\pi/4), (0, 6\pi/4), (2\pi/3,$ and one constant solution. Since the Hilbert-space dimension is 6, one solution is missing. See Figure 7 for the real solutions of the Bethe equations. Let us solve the eigenvalue equation directly and compare with the Bethe-ansatz solutions.

There are variables $\boldsymbol{f} = (f(1,2), f(1,3), f(1,4), f(2,3), f(2,4), f(3,4))^T$, and the eigenvalue equation is

$$\begin{cases} f(2,4) + f(1,3) + 2f(1,2) = Ef(1,2), \\ f(3,4) + f(1,2) + f(2,3) + f(1,4) = Ef(1,3), \\ f(1,3) + f(2,4) + 2f(1,4) = Ef(1,4), \\ f(1,3) + f(2,4) + 2f(2,3) = Ef(2,3), \\ f(1,4) + f(2,3) + f(3,4) + f(1,2) = Ef(2,4), \\ f(2,4) + f(1,3) + 2f(3,4) = Ef(3,4), \end{cases} \tag{110}$$

that is,

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{pmatrix} \boldsymbol{f} = E\boldsymbol{f}. \tag{111}$$

Solving this, we obtain the following six solutions:

$$\boldsymbol{f} = (1,1,1,1,1,1)^T, \quad E = 4, \tag{112}$$
$$\boldsymbol{f} = (1,-2,1,1,-2,1)^T, \quad E = -2, \tag{113}$$
$$\boldsymbol{f} = (0,-1,0,0,1,0)^T, \quad E = 0, \tag{114}$$
$$\boldsymbol{f} = (-1,0,0,0,0,1)^T, \quad E = 2, \tag{115}$$
$$\boldsymbol{f} = (-1,0,0,1,0,0)^T, \quad E = 2, \tag{116}$$
$$\boldsymbol{f} = (-1,0,1,0,0,0)^T, \quad E = 2. \tag{117}$$

On the other hand,

$$\text{constant solution}: \quad \boldsymbol{f} = (1,1,1,1,1,1), \quad E = 4, \tag{118}$$
$$(k_1, k_2) = (\frac{2\pi}{3}, \frac{4\pi}{3}): \quad \boldsymbol{f} = (1,-2,1,1,-2,1), \quad E = -2, \tag{119}$$
$$(k_1, k_2) = (0, \pi): \quad \boldsymbol{f} = (0,-1,0,0,1,0), \quad E = 0, \tag{120}$$
$$(k_1, k_2) = (0, \frac{\pi}{2}): \quad \boldsymbol{f} = (1,0,-i,i,0,-1), \quad E = 2, \tag{121}$$
$$(k_1, k_2) = (0, \frac{3\pi}{2}): \quad \boldsymbol{f} = (1,0,i,-i,0,-1), \quad E = 2 \tag{122}$$

Thus one solution with $E = 2$ is indeed missing.

In general, when the number of sites is even, the following solution of the eigenvalue equation exists. [2]

$$|\psi\rangle = \sum_{j=1}^{N} (-1)^j \sigma_j^- \sigma_{j+1}^- |0\rangle . \tag{123}$$

Indeed, substituting the above $|\psi\rangle$ into the eigenvalue equation confirms that it is a solution with eigenvalue $E = 2$. The missing solution seems to be due to having neglected the case in which the denominator in (56) vanishes, but I am not fully satisfied with this explanation. For example, [5] contains a discussion. I record a note on this in the next subsection.

1.2.12 "Singular Solutions" of the Bethe Equations

$$e^{ik_j} = \frac{\lambda_j + i}{\lambda_j - i}, \quad j = 1, 2, \tag{124}$$

and introduce λ_j. Then

$$1 + e^{i(k_1+k_2)} - 2e^{ik_1} = \frac{-2i(\lambda_2 - \lambda_1 - 2i)}{(\lambda_1 - i)(\lambda_2 - i)}, \tag{125}$$

$$1 + e^{i(k_1+k_2)} - 2e^{ik_2} = \frac{-2i(\lambda_1 - \lambda_2 - 2i)}{(\lambda_1 - i)(\lambda_2 - i)} \tag{126}$$

Using these identities, the Bethe equations become

$$\left(\frac{\lambda_1 + i}{\lambda_1 - i}\right)^N = -\frac{\lambda_2 - \lambda_1 - 2i}{\lambda_1 - \lambda_2 - 2i}, \tag{127}$$

$$\left(\frac{\lambda_2 + i}{\lambda_2 - i}\right)^N = -\frac{\lambda_1 - \lambda_2 - 2i}{\lambda_2 - \lambda_1 - 2i} \tag{128}$$

The case in which the denominator vanishes is singular, so after clearing denominators,

$$(\lambda_1 + i)^N(\lambda_1 - \lambda_2 - 2i) = -(\lambda_1 - i)^N(\lambda_2 - \lambda_1 - 2i), \tag{129}$$

$$(\lambda_2 + i)^N(\lambda_2 - \lambda_1 - 2i) = -(\lambda_2 - i)^N(\lambda_1 - \lambda_2 - 2i) \tag{130}$$

we obtain the solution

$$\lambda_1 = i, \lambda_2 = -i \tag{131}$$

It is known that, under a suitable regularization, the corresponding eigenvalue and eigenvector give $|\psi\rangle$ and $E = 2$ [5].

1.2.13 The Solutions Are Given by Complex-Conjugate Pairs

From the symmetry of the Bethe equations under complex conjugation, if $\{k_1, k_2\}$ is a solution, then $\{k_1^*, k_2^*\}$ is also a solution. In fact, as sets one has $\{k_1, k_2\} = \{k_1^*, k_2^*\}$. For general M, this was proved using the transfer matrix in [7]. Also, [6] states only that, by the symmetry of the Bethe equations under complex conjugation, one may take $k_2 = k_1^*$ for complex solutions. Is that really sufficient?

[2] I learned this from Hosho Katsura. I thank him.

For $M=2$, one can prove this as follows using the fact that momentum and energy are real. [3] Since k_1+k_2 is the total momentum, it is real; in particular, $\mathrm{Im}\,k_1+\mathrm{Im}\,k_2=0$. When $\mathrm{Im}\,k_1=-\mathrm{Im}\,k_2\neq 0$, we want to show $\mathrm{Re}\,k_1=\mathrm{Re}\,k_2$. The energy is

$$E_{\boldsymbol{k}}=2\cos k_1+2\cos k_2=4\cos\frac{k_1+k_2}{2}\cos\frac{k_1-k_2}{2} \tag{132}$$

and its imaginary part is

$$\mathrm{Im}\,E_{\boldsymbol{k}}=-4\cos\frac{k_1+k_2}{2}\sin\frac{\mathrm{Re}\,(k_1-k_2)}{2}\sinh\frac{\mathrm{Im}\,(k_1-k_2)}{2}. \tag{133}$$

Since the Hamiltonian is Hermitian, $\mathrm{Im}\,E_{\boldsymbol{k}}=0$. Under the assumption $\mathrm{Im}\,k_1=-\mathrm{Im}\,k_2\neq 0$, this implies either $k_1+k_2=\pi \mod 2\pi$, or $\mathrm{Re}\,(k_1-k_2)=0$. Thus the claim is proved when $k_1+k_2\neq\pi \mod 2\pi$.

Now consider the case $k_1+k_2=\pi$. In this case the Bethe equations are

$$e^{ik_1N}=-\frac{1+e^{i(k_1+k_2)}-2e^{ik_1}}{1+e^{i(k_1+k_2)}-2e^{ik_2}}=e^{2ik_1}, \tag{134}$$

$$e^{ik_2N}=-\frac{1+e^{i(k_1+k_2)}-2e^{ik_2}}{1+e^{i(k_1+k_2)}-2e^{ik_1}}=e^{2ik_2}, \tag{135}$$

Solutions exist when N is even, and they are

$$(k_1,k_2)=(\frac{2\pi n}{N-2},\pi-\frac{2\pi n}{N-2}),\quad n=0,\dots,N-3 \tag{136}$$

Thus, in the case $k_1+k_2=\pi$, there are no complex solutions.

1.2.14 Comparison with Existing Results

[6] points out the shortage of M_N complex-conjugate solutions discussed in Sections 1.2.7 and 1.2.8, together with the accompanying appearance of real solutions with close wavenumbers. The calculation in this note agrees with the calculation in [6].

1.2.15 Spectrum of Complex-Conjugate Solutions

Let us compute the spectrum of the complex-conjugate solutions for large N. Ignoring the $2M\sim\frac{2\sqrt{N}}{\pi}$ exceptional solutions, and further taking $(u,v)=(u_n,-\log\cos u_n)$ as an approximate solution, the solution (k_1,k_2) is roughly

$$k_1=k-i\log\cos k, \tag{137}$$

$$k_2=k+i\log\cos k, \tag{138}$$

$$k=\frac{\pi n}{N},\quad n=-\frac{-N}{2},-\frac{N}{n}+1,\dots,\frac{N}{2}, \tag{139}$$

The energy eigenvalue is

$$E(k_1,k_2)=2\cos k_1+2\cos k_2=e^{ik_1}+e^{-ik_1}+e^{ik_2}+e^{-ik_2} \tag{140}$$

$$=e^{ik}\cos k+e^{-ik}(\cos k)^{-1}+e^{ik}(\cos k)^{-1}+e^{-ik}\cos k \tag{141}$$

$$=2\cos^2 k+2=\cos 2k+4,\quad -\frac{\pi}{2}<k<\frac{\pi}{2}. \tag{142}$$

[3] I also learned this proof from Hosho Katsura.

1.2.16 Backup Calculation

Let us consider complex solutions. I follow the argument in [1]. First rewrite the Bethe equations slightly.

$$e^{i\varphi_{\boldsymbol{k}}} = -\frac{\cos\frac{k_1+k_2}{2} - e^{-i(k_2-k_1)/2}}{\cos\frac{k_1+k_2}{2} - e^{i(k_2-k_1)/2}} = -\frac{\cos\frac{k_1+k_2}{2} - \cos\frac{k_1-k_2}{2} - i\sin\frac{k_1-k_2}{2}}{\cos\frac{k_1+k_2}{2} - \cos\frac{k_1-k_2}{2} + i\sin\frac{k_1-k_2}{2}} \tag{143}$$

From this,

$$\cot\frac{\varphi_{\boldsymbol{k}}}{2} = \frac{1}{\tan\frac{\varphi_{\boldsymbol{k}}}{2}} = \frac{\sin\varphi_{\boldsymbol{k}}}{1-\cos\varphi_{\boldsymbol{k}}} = \frac{1}{i}\frac{e^{i\varphi_{\boldsymbol{k}}} - e^{-i\varphi_{\boldsymbol{k}}}}{2 - e^{i\varphi_{\boldsymbol{k}}} - e^{-i\varphi_{\boldsymbol{k}}}} = \frac{\sin\frac{k_1-k_2}{2}}{\cos\frac{k_1+k_2}{2} - \cos\frac{k_1-k_2}{2}} = \frac{1}{2}(\cot\frac{k_1}{2} - \cot\frac{k_2}{2}), \tag{144}$$

namely,

$$2\cot\frac{\varphi_{\boldsymbol{k}}}{2} = \cot\frac{k_1}{2} - \cot\frac{k_2}{2} \tag{145}$$

$$k_1 = u + iv, \quad k_2 = u - iv \tag{146}$$

With this parametrization, since the total momentum $k_1 + k_2$ is real, u is real. The variable v is complex in general.

$$\begin{aligned}
\cot\frac{k_1}{2} &= \frac{\cos\frac{u+iv}{2}}{\sin\frac{u+iv}{2}} = \frac{\cos\frac{u}{2}\cos\frac{iv}{2} - \sin\frac{u}{2}\sin\frac{iv}{2}}{\sin\frac{u}{2}\cos\frac{iv}{2} + \cos\frac{u}{2}\sin\frac{iv}{2}} = \frac{\cos\frac{u}{2}\cosh\frac{v}{2} - i\sin\frac{u}{2}\sinh\frac{v}{2}}{\sin\frac{u}{2}\cosh\frac{v}{2} + i\cos\frac{u}{2}\sinh\frac{v}{2}} &(147)\\
&= \frac{(\cos\frac{u}{2}\cosh\frac{v}{2} - i\sin\frac{u}{2}\sinh\frac{v}{2})(\sin\frac{u}{2}\cosh\frac{v}{2} - i\cos\frac{u}{2}\sinh\frac{v}{2})}{\sin^2\frac{u}{2}\cosh^2\frac{v}{2} + \cos^2\frac{u}{2}\sinh^2\frac{v}{2}} &(148)\\
&= \frac{\cos\frac{u}{2}\sin\frac{u}{2} - i\cosh\frac{v}{2}\sinh\frac{v}{2}}{\sin^2\frac{u}{2}\cosh^2\frac{v}{2} + \cos^2\frac{u}{2}\sinh^2\frac{v}{2}} &(149)\\
&= \frac{2\sin u - 2i\sinh v}{(1-\cos u)(1+\cosh v) + (1+\cos u)(\cosh v - 1)} &(150)\\
&= \frac{\sin u - i\sinh v}{\cosh v - \cos u}. &(151)
\end{aligned}$$

Replacing v by $-v$, we get

$$\cot\frac{k_2}{2} = \frac{\sin u + i\sinh v}{\cosh v - \cos u}. \tag{152}$$

Although v is complex in general, in the following I take v to be real. Thus k_1, k_2 are assumed to be complex conjugates. It is enough to consider $v > 0$.

$$\varphi_{\boldsymbol{k}} = \psi_{\boldsymbol{k}} + i\chi_{\boldsymbol{k}} \tag{153}$$

and separate real and imaginary parts. Then the Bethe equations become

$$\psi_{\boldsymbol{k}} \equiv -\psi_{\boldsymbol{k}} \mod 2\pi, \tag{154}$$

$$Nv = \chi_{\boldsymbol{k}} \tag{155}$$

2 Open Boundary Conditions

Since the Bethe ansatz is obtained by connecting free-field solutions for $|x_2 - x_1| > 1$ to the region $|x_2 - x_1| = 1$, one might naively expect it to apply also to open boundary conditions. For open boundary conditions, the Hamiltonian is

$$H = \sum_{j=1}^{N-1} (\sigma_{j+1}^- \sigma_j^+ + \sigma_j^- \sigma_{j+1}^+ + 2 \frac{1-\sigma_j^z}{2} \frac{1-\sigma_{j+1}^z}{2}). \tag{156}$$

Let us examine the structure of the eigenvalue equation

$$H \left| f \right\rangle_M = E \left| f \right\rangle_M \tag{157}$$

2.1 $M = 1$

For $M = 1$, the third term drops out, and

$$H \left| f \right\rangle_1 = \sum_{j=1}^{N-1} \sum_{x=1}^{N} (\delta_{jx} \sigma_{j+1}^- + \delta_{j+1,x} \sigma_j^-) f(x) \left| 0 \right\rangle \tag{158}$$

$$= \sum_{j=1}^{N-1} (f(j) \sigma_{j+1}^- + f(j+1) \sigma_j^-) \left| 0 \right\rangle \tag{159}$$

$$= \sum_{j=2}^{N} f(j-1) \sigma_j^- \left| 0 \right\rangle + \sum_{j=1}^{N-1} f(j+1) \sigma_j^- \left| 0 \right\rangle . \tag{160}$$

This gives the eigenvalue equation

$$\begin{cases} f(j-1) + f(j+1) = E f(j) \quad j = 2, \ldots, N-1, \\ f(2) = E f(1), \\ f(N-1) = E f(N), \end{cases} \tag{161}$$

If $f(0)$ and $f(N+1)$ are added as variables, this is equivalent to

$$f(j-1) + f(j+1) = E f(j), \qquad j = 1, \ldots, N, \tag{162}$$

$$f(0) = f(N+1) = 0. \tag{163}$$

The N linearly independent solutions are

$$f_k(x) = \sin \frac{\pi p x}{N+1}, \quad p = 1, \ldots, N. \tag{164}$$

2.2 $M = 2$

For the hopping terms, when $x_1 \neq x_2$,

$$\sigma_{j+1}^- \sigma_j^+ \sigma_{x_1}^- \sigma_{x_2}^- \left| 0 \right\rangle = \delta_{j x_1} \sigma_{x_1+1}^- \sigma_{x_2}^- \left| 0 \right\rangle + \delta_{j x_2} \sigma_{x_1}^- \sigma_{x_2+1}^- \left| 0 \right\rangle , \tag{165}$$

$$\sigma_j^- \sigma_{j+1}^+ \sigma_{x_1}^- \sigma_{x_2}^- \left| 0 \right\rangle = \delta_{j+1, x_1} \sigma_{x_1-1}^- \sigma_{x_2}^- \left| 0 \right\rangle + \delta_{j+1, x_2} \sigma_{x_1}^- \sigma_{x_2-1}^- \left| 0 \right\rangle , \tag{166}$$

Notice these relations. When two lowering operators act on the same site, set $\sigma_x^- \sigma_x^- = 0$. From this,

$$\sum_{j=1}^{N-1} \sigma_{j+1}^- \sigma_j^+ \left| f \right\rangle_2 = \sum_{1 \leq x_1 < x_2 \leq N} \sum_{j=1}^{N-1} f(x_1, x_2) (\delta_{j x_1} \sigma_{x_1+1}^- \sigma_{x_2}^- \left| 0 \right\rangle + \delta_{j x_2} \sigma_{x_1}^- \sigma_{x_2+1}^- \left| 0 \right\rangle) \tag{167}$$

$$= \sum_{1 \leq x_1 < x_2 - 1 \leq N-1} f(x_1, x_2) \sigma_{x_1+1}^- \sigma_{x_2}^- \left| 0 \right\rangle + \sum_{1 \leq x_1 < x_2 \leq N-1} f(x_1, x_2) \sigma_{x_1}^- \sigma_{x_2+1}^- \left| 0 \right\rangle \tag{168}$$

$$= \sum_{2 \leq x_1 < x_2 \leq N} f(x_1 - 1, x_2) \sigma_{x_1}^- \sigma_{x_2}^- \left| 0 \right\rangle + \sum_{1 \leq x_1 < x_2 - 1 \leq N-1} f(x_1, x_2 - 1) \sigma_{x_1}^- \sigma_{x_2}^- \left| 0 \right\rangle , \tag{169}$$

$$\sum_{j=1}^{N-1} \sigma_j^- \sigma_{j+1}^+ |f\rangle_2 = \sum_{1\le x_1<x_2\le N} \sum_{j=1}^{N-1} f(x_1,x_2)(\delta_{j+1,x_1}\sigma_{x_1-1}^- \sigma_{x_2}^- |0\rangle + \delta_{j+1,x_2}\sigma_{x_1}^-\sigma_{x_2-1}^- |0\rangle) \tag{170}$$

$$= \sum_{2\le x_1<x_2\le N} f(x_1,x_2)\sigma_{x_1-1}^-\sigma_{x_2}^- |0\rangle + \sum_{1\le x_1<x_2-1\le N-1} f(x_1,x_2)\sigma_{x_1}^-\sigma_{x_2-1}^- |0\rangle \tag{171}$$

$$= \sum_{1\le x_1<x_2-1\le N-1} f(x_1+1,x_2)\sigma_{x_1}^-\sigma_{x_2}^- |0\rangle + \sum_{1\le x_1<x_2\le N-1} f(x_1,x_2+1)\sigma_{x_1}^-\sigma_{x_2}^- |0\rangle\,. \tag{172}$$

For the interaction term,

$$\sum_{j=1}^{N-1} \frac{1-\sigma_j^z}{2}\frac{1-\sigma_{j+1}^z}{2}\sigma_{x_1}^-\sigma_{x_2}^- |0\rangle = \delta_{x_1+1,x_2}\sigma_{x_1}^-\sigma_{x_2}^- |0\rangle \tag{173}$$

and therefore

$$\sum_{1\le x_1<x_2\le N} \sum_{j=1}^{N-1} 2\frac{1-\sigma_j^z}{2}\frac{1-\sigma_{j+1}^z}{2} |f\rangle_2 = \sum_{1\le x_1<x_2\le N} f(x_1,x_2)2\delta_{x_1+1,x_2}\sigma_{x_1}^-\sigma_{x_2}^- |0\rangle \tag{174}$$

$$= \sum_{x=1}^{N-1} f(x,x+1)2\sigma_x^-\sigma_{x+1}^- |0\rangle\,. \tag{175}$$

Thus the action of the Hamiltonian on the wave function $|f\rangle_2$ is

$$H|f\rangle_2 = \sum_{2\le x_1<x_2\le N} f(x_1-1,x_2)\sigma_{x_1}^-\sigma_{x_2}^- |0\rangle + \sum_{1\le x_1<x_2-1\le N-1} f(x_1,x_2-1)\sigma_{x_1}^-\sigma_{x_2}^- |0\rangle \tag{176}$$

$$+ \sum_{1\le x_1<x_2-1\le N-1} f(x_1+1,x_2)\sigma_{x_1}^-\sigma_{x_2}^- |0\rangle + \sum_{1\le x_1<x_2\le N-1} f(x_1,x_2+1)\sigma_{x_1}^-\sigma_{x_2}^- |0\rangle \tag{177}$$

$$+ \sum_{x=1}^{N-1} f(x,x+1)2\sigma_x^-\sigma_{x+1}^- |0\rangle \tag{178}$$

The eigenvalue equation is

- For $2\le x_1$, $x_1+1\le x_2\le N-1$ (yellow),

$$(A): \quad f(x_1-1,x_2)+f(x_1,x_2-1)+f(x_1+1,x_2)+f(x_1,x_2+1) = Ef(x_1,x_2). \tag{179}$$

- For $x_1=1$, $3\le x_2\le N-1$ (cyan),

$$(B): \quad f(1,x_2-1)+f(2,x_2)+f(1,x_2+1) = Ef(1,x_2). \tag{180}$$

- For $2\le x_1\le N-2$, $x_2=N$ (orange),

$$(C): \quad f(x_1-1,N)+f(x_1,N-1)+f(x_1+1,N) = Ef(x_1,N). \tag{181}$$

- For $x_1=x$, $x_2=x+1$, $2\le x\le N-2$ (green),

$$(D): \quad f(x-1,x+1)+f(x,x+2)+2f(x,x+1) = Ef(x,x+1). \tag{182}$$

- For $x_1=1, x_2=2$ (blue),

$$(E): \quad f(1,3)+2f(1,2) = Ef(1,2). \tag{183}$$

- For $x_1=N-1, x_2=N$ (red),

$$(F): \quad f(N-2,N)+2f(N-1,N) = Ef(N-1,N). \tag{184}$$

- For $x_1 = 1, x_2 = N$ (pink),

$$(G): \quad 0 = Ef(1, N). \tag{185}$$

As in the periodic-boundary case, it is convenient to extend the domain $1 \leq x_1 < x_2 \leq N$ of $f(x_1, x_2)$ and introduce boundary conditions for the extra degrees of freedom. Introduce the following $2N + 1$ variables:

$$f(0, x_2), \quad x_2 = 1, \ldots, N+1, \tag{186}$$
$$f(x_1, N+1), \quad x_1 = 0, \ldots, N, \tag{187}$$

and impose the following boundary conditions:

$$\text{(B.C.)} \quad \begin{cases} f(0, x_2) = 0, & x_2 = 1, \ldots, N+1, \\ f(x_1, N+1) = 0, & x_1 = 0, \ldots, N. \end{cases} \tag{188}$$

Then the equations (A,B,C) can be combined as

$$\text{(Eq.1)}: \quad f(x_1 - 1, x_2) + f(x_1, x_2 - 1) + f(x_1 + 1, x_2) + f(x_1, x_2 + 1) = Ef(x_1, x_2), \quad |x_2 - x_1| > 1, \tag{189}$$

Moreover, (D,E,F,G) can also be rewritten as

$$\text{(Eq.2)}: \quad f(x-1, x+1) + f(x, x+2) + 2f(x, x+1) = Ef(x, x+1), \quad (x = 1, \ldots, N-1), \tag{190}$$

We want to solve the eigenvalue equations (Eq.1, Eq.2, G) under the boundary conditions (B.C.). (Eq.1) has the following eight free-field solutions:

$$e^{ik_1x_1}e^{ik_2x_2}, e^{ik_1x_2}e^{ik_2x_1}, e^{-ik_1x_1}e^{ik_2x_2}, e^{-ik_1x_2}e^{ik_2x_1}, \tag{191}$$
$$e^{ik_1x_1}e^{-ik_2x_2}, e^{ik_1x_2}e^{-ik_2x_1}, e^{-ik_1x_1}e^{-ik_2x_2}, e^{-ik_1x_2}e^{-ik_2x_1}, \tag{192}$$
$$E_{\boldsymbol{k}} = 2\cos k_1 + 2\cos k_2 \tag{193}$$

Thus write the solution as

$$g_{\boldsymbol{k}}(x_1, x_2) = c_1^{++}e^{ik_1x_1}e^{ik_2x_2} + c_2^{++}e^{ik_1x_2}e^{ik_2x_1} + c_1^{-+}e^{-ik_1x_1}e^{ik_2x_2} + c_2^{-+}e^{-ik_1x_2}e^{ik_2x_1} \tag{194}$$
$$+ c_1^{+-}e^{ik_1x_1}e^{-ik_2x_2} + c_2^{+-}e^{ik_1x_2}e^{-ik_2x_1} + c_1^{--}e^{-ik_1x_1}e^{-ik_2x_2} + c_2^{--}e^{-ik_1x_2}e^{-ik_2x_1} \tag{195}$$

The function $g_{\boldsymbol{k}}(x_1, x_2)$ again satisfies the relation

$$g_{\boldsymbol{k}}(x-1, x+1) + g_{\boldsymbol{k}}(x, x) + g_{\boldsymbol{k}}(x+1, x+1) + g_{\boldsymbol{k}}(x, x+2) = E_{\boldsymbol{k}}g_{\boldsymbol{k}}(x, x+1) \tag{196}$$

so substituting $g_{\boldsymbol{k}}(x)$ into (Eq.2), the condition for $g_{\boldsymbol{k}}(x_1, x_2)$ to be a solution is

$$g_{\boldsymbol{k}}(x, x) + g_{\boldsymbol{k}}(x+1, x+1) - 2g_{\boldsymbol{k}}(x, x+1) = 0 \tag{197}$$

This equation has a solution, and for $c_1^{\sigma\sigma'}, c_2^{\sigma\sigma'}$ we obtain

$$\frac{c_1^{\sigma\sigma'}}{c_2^{\sigma\sigma'}} = -\frac{1 + e^{i(\sigma k_1 + \sigma' k_2)} - 2e^{i\sigma k_1}}{1 + e^{i(\sigma k_1 + \sigma' k_2)} - 2e^{i\sigma' k_2}} =: e^{i\varphi_{\sigma\sigma'}}, \quad \sigma, \sigma' \in \{+, -\}, \tag{198}$$

Since the equation is linear, we are taking a linear combination of the solutions for each $(\pm k_1, \pm k_2)$. Noting that $e^{i\varphi_{--}} = e^{-i\varphi_{++}}$ and $e^{i\varphi_{+-}} = e^{-i\varphi_{-+}}$, the ansatz solution satisfying (Eq.1, Eq.2) is, with coefficients a, b, c, d,

$$g_{\boldsymbol{k}}(x_1, x_2) = a(e^{i(k_1x_1 + k_2x_2 + \frac{\alpha}{2})} + e^{i(k_1x_2 + k_2x_1 - \frac{\alpha}{2})}) + b(e^{-i(k_1x_1 + k_2x_2 + \frac{\alpha}{2})} + e^{-i(k_1x_2 + k_2x_1 - \frac{\alpha}{2})}) \tag{199}$$
$$+ c(e^{i(k_1x_1 - k_2x_2 + \frac{\beta}{2})} + e^{i(k_1x_2 - k_2x_1 - \frac{\beta}{2})}) + d(e^{-i(k_1x_1 - k_2x_2 + \frac{\beta}{2})} + e^{-i(k_1x_2 - k_2x_1 - \frac{\beta}{2})}) \tag{200}$$

where

$$e^{i\alpha} = -\frac{1+e^{i(k_1+k_2)}-2e^{ik_1}}{1+e^{i(k_1+k_2)}-2e^{ik_2}}, \tag{201}$$

$$e^{i\beta} = -\frac{1+e^{i(k_1-k_2)}-2e^{ik_1}}{1+e^{i(k_1-k_2)}-2e^{-ik_2}} = -\frac{e^{ik_1}+e^{ik_2}-2e^{i(k_1+k_2)}}{e^{ik_1}+e^{ik_2}-2} \tag{202}$$

Next solve the boundary conditions (B.C.). From $f(0,x_2)=0$,

$$a(e^{i(k_2x_2+\frac{\alpha}{2})}+e^{i(k_1x_2-\frac{\alpha}{2})})+b(e^{-i(k_2x_2+\frac{\alpha}{2})}+e^{-i(k_1x_2-\frac{\alpha}{2})}) \tag{203}$$

$$+c(e^{i(-k_2x_2+\frac{\beta}{2})}+e^{i(k_1x_2-\frac{\beta}{2})})+d(e^{-i(-k_2x_2+\frac{\beta}{2})}+e^{-i(k_1x_2-\frac{\beta}{2})}) \tag{204}$$

$$=(ae^{-i\alpha/2}+ce^{-i\beta/2})e^{ik_1x_2}+(be^{i\alpha/2}+de^{i\beta/2})e^{-ik_1x_2} \tag{205}$$

$$+(ae^{i\alpha/2}+de^{-i\beta/2})e^{ik_2x_2}+(be^{-i\alpha/2}+ce^{i\beta/2})e^{-ik_2x_2}. \tag{206}$$

Thus, assuming $k_1 \neq k_2$,

$$\begin{cases} ae^{-i\alpha/2}+ce^{-i\beta/2}=0, \\ be^{i\alpha/2}+de^{i\beta/2}=0, \\ ae^{i\alpha/2}+de^{-i\beta/2}=0, \\ be^{-i\alpha/2}+ce^{i\beta/2}=0, \end{cases} \tag{207}$$

Solving these equations gives

$$b=ae^{i\beta}, \quad c=-ae^{-i(\alpha-\beta)/2}, \quad d=-ae^{i(\alpha+\beta)/2} \tag{208}$$

and therefore the ansatz solution is determined as

$$g_{\boldsymbol{k}}(x_1,x_2)=(e^{i(k_1x_1+k_2x_2+\frac{\alpha}{2})}+e^{i(k_1x_2+k_2x_1-\frac{\alpha}{2})})e^{-i\beta/2}+(e^{-i(k_1x_1+k_2x_2+\frac{\alpha}{2})}+e^{-i(k_1x_2+k_2x_1-\frac{\alpha}{2})})e^{i\beta/2} \tag{209}$$

$$-(e^{i(k_1x_1-k_2x_2+\frac{\beta}{2})}+e^{i(k_1x_2-k_2x_1-\frac{\beta}{2})})e^{-i\alpha/2}-(e^{-i(k_1x_1-k_2x_2+\frac{\beta}{2})}+e^{-i(k_1x_2-k_2x_1-\frac{\beta}{2})})e^{i\alpha/2}. \tag{210}$$

The condition that the boundary condition $f(x_1,N+1)=0$ hold for arbitrary x_1 gives equations for k_1,k_2. Set $L=N+1$. Then

$$f(x_1,L)=(e^{i(k_1x_1+k_2L+\frac{\alpha}{2})}+e^{i(k_1L+k_2x_1-\frac{\alpha}{2})})e^{-i\beta/2}+(e^{-i(k_1x_1+k_2L+\frac{\alpha}{2})}+e^{-i(k_1L+k_2x_1-\frac{\alpha}{2})})e^{i\beta/2} \tag{211}$$

$$-(e^{i(k_1x_1-k_2L+\frac{\beta}{2})}+e^{i(k_1L-k_2x_1-\frac{\beta}{2})})e^{-i\alpha/2}-(e^{-i(k_1x_1-k_2L+\frac{\beta}{2})}+e^{-i(k_1L-k_2x_1-\frac{\beta}{2})})e^{i\alpha/2} \tag{212}$$

$$=(e^{i(k_2L+\frac{\alpha}{2})}e^{-i\beta/2}-e^{i(-k_2L+\frac{\beta}{2})}e^{-i\alpha/2})e^{ik_1x_1}+(e^{-i(k_2L+\frac{\alpha}{2})}e^{i\beta/2}-e^{-i(-k_2L+\frac{\beta}{2})}e^{i\alpha/2})e^{-ik_1x_1} \tag{213}$$

$$+(e^{i(k_1L-\frac{\alpha}{2})}e^{-i\beta/2}-e^{-i(k_1L-\frac{\beta}{2})}e^{i\alpha/2})e^{ik_2x_1}+(e^{-i(k_1L-\frac{\alpha}{2})}e^{i\beta/2}-e^{i(k_1L-\frac{\beta}{2})}e^{-i\alpha/2})e^{-ik_2x_1} \tag{214}$$

$$=2i\sin(k_2L+\frac{\alpha-\beta}{2})e^{ik_1x_1}-2i\sin(k_2L+\frac{\alpha-\beta}{2})e^{-ik_1x_1} \tag{215}$$

$$+2i\sin(k_1L-\frac{\alpha+\beta}{2})e^{ik_2x_1}-2i\sin(k_1L-\frac{\alpha+\beta}{2})e^{-ik_2x_1} \tag{216}$$

$$=-4\sin(k_2L+\frac{\alpha-\beta}{2})\sin(k_1x_1)-4\sin(k_1L-\frac{\alpha+\beta}{2})\sin(k_2x_1). \tag{217}$$

Thus, assuming $k_1 \neq k_2$, the boundary condition gives

$$\sin(k_2L+\frac{\alpha-\beta}{2})=\sin(k_1L-\frac{\alpha+\beta}{2})=0, \tag{218}$$

or equivalently

$$e^{2ik_2(N+1)}=e^{-i\alpha}e^{i\beta}=\frac{1+e^{i(k_1+k_2)}-2e^{ik_2}}{1+e^{i(k_1+k_2)}-2e^{ik_1}}\times\frac{e^{ik_1}+e^{ik_2}-2e^{i(k_1+k_2)}}{e^{ik_1}+e^{ik_2}-2}, \tag{219}$$

$$e^{2ik_1(N+1)}=e^{i\alpha}e^{i\beta}=\frac{1+e^{i(k_1+k_2)}-2e^{ik_1}}{1+e^{i(k_1+k_2)}-2e^{ik_2}}\times\frac{e^{ik_1}+e^{ik_2}-2e^{i(k_1+k_2)}}{e^{ik_1}+e^{ik_2}-2} \tag{220}$$

These correspond to the Bethe equations for open boundary conditions. Compare with existing results.

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