A Calculation Note on the Chern–Simons 3-Form

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The Chern–Simons (CS) 3-form

$$\frac{1}{2!}\left(\frac{i}{2\pi}\right)^2 \mathrm{tr}\left[AdA+\frac{2}{3}A^3\right] = -\frac{1}{8\pi^2}\mathrm{tr}\left[AdA+\frac{2}{3}A^3\right] \tag{1}$$

is not gauge invariant. Let A_i denote the gauge field on a patch U_i. A gauge transformation is

$$A_j = v_{ij}^{-1}(A_i+d)v_{ij}, \quad v_{ij}\in U(N). \tag{2}$$

We compute the change of the CS 3-form under this gauge transformation. Since

$$v^{-1}(A+d)v = v^{-1}(A+dv\,v^{-1})v, \tag{3}$$

we get

$$\begin{aligned}
&\delta\mathrm{tr}\left[AdA+\frac{2}{3}A^3\right]\\
&=\mathrm{tr}\left[v^{-1}(A+dv\,v^{-1})vd\{v^{-1}(A+dv\,v^{-1})v\}\frac{2}{3}\{v^{-1}(A+dv\,v^{-1})v\}^3\right]-\mathrm{tr}\left[AdA+\frac{2}{3}A^3\right]\\
&=\mathrm{tr}\left[(A+dv\,v^{-1})(-dv\,v^{-1}A+dA-Adv\,v^{-1}-dv\,v^{-1}dv\,v^{-1})\frac{2}{3}(A+dv\,v^{-1})^3\right]\\
&\quad-\mathrm{tr}\left[AdA+\frac{2}{3}A^3\right].
\end{aligned} \tag{4}$$

Put

$$X = dv\,v^{-1}. \tag{5}$$

Then

$$\begin{aligned}
&\delta\mathrm{tr}\left[AdA+\frac{2}{3}A^3\right]\\
&=\mathrm{tr}\left[(A+X)(-XA+dA-AX-X^2)+\frac{2}{3}(A+X)^3\right]-\mathrm{tr}\left[AdA+\frac{2}{3}A^3\right]\\
&=\mathrm{tr}\Big[-AXA+AdA-A^2X-AX^2-X^2A+XdA-XAX-X^3\\
&\quad+\frac{2}{3}(A^3+A^2X+AXA+AX^2+XA^2+XAX+X^2A+X^3)\Big]-\mathrm{tr}\left[AdA+\frac{2}{3}A^3\right]\\
&=\mathrm{tr}\left[XdA-AX^2-\frac{1}{3}X^3\right].
\end{aligned} \tag{6}$$

Here we used the cyclicity

$$\mathrm{tr}\,[XYZ]=\mathrm{tr}\,[X_\mu Y_\nu Z_\rho]\epsilon_{\mu\nu\rho}=\mathrm{tr}\,[Y_\mu Z_\nu X_\rho]\epsilon_{\mu\nu\rho}=\mathrm{tr}\,[YZX]. \tag{7}$$

Since

$$X^2 = dv\, v^{-1} dv\, v^{-1} = -dv\, dv^{-1} = d(dv\, v^{-1}) = dX, \tag{8}$$

we have

$$\mathrm{tr}\,[XdA - AdX] = \mathrm{tr}\,[XdA - dXA] = -d\mathrm{tr}\,[XA]. \tag{9}$$

Equivalently, one may note that

$$\mathrm{tr}\,[XdA - dXA] = \mathrm{tr}\,[dAX - AdX] = d\mathrm{tr}\,[AX] = -d\mathrm{tr}\,[XA]. \tag{10}$$

Therefore

$$\delta\mathrm{tr}\,\left[AdA + \frac{2}{3}A^3\right] = -d\mathrm{tr}\,[dv\, v^{-1}A] - \frac{1}{3}\mathrm{tr}\,[dv\, v^{-1}]^3. \tag{11}$$

The first term is a total derivative. The second term is the volume form that computes the degree of a map to $U(N)$.

1 $SU(2)$ case

The $SU(2)$ case can be analyzed by specializing the above formula to $v \in SU(2)$.