Canonical Adiabatic Pump: 1D Class D

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For complex fermions $a_j^\dagger/a_j$, introduce Majorana fermions by

$$c_{\sigma,2j-1} = a_{\sigma,j} + a_{\sigma,j}^\dagger, \quad c_{\sigma,2j} = \frac{a_{\sigma,j} - a_{\sigma,j}^\dagger}{i}. \tag{1}$$

The model is

$$H(\theta) = \begin{cases} H_I(\theta), & \theta \in [0,\pi], \\ H_{II}(\theta), & \theta \in [\pi, 2\pi], \end{cases} \tag{2}$$

with

$$H_I(\theta) = \frac{i}{2}\sum_j [\cos\theta(c_{\uparrow,2j-1}c_{\uparrow,2j} + c_{\downarrow,2j-1}c_{\downarrow,2j}) \tag{3}$$

$$+\sin\theta(c_{\uparrow,2j-1}c_{\downarrow,2j-1} + c_{\downarrow,2j}c_{\uparrow,2j})], \tag{4}$$

$$H_{II}(\theta) = \frac{i}{2}\sum_j [\cos\theta(c_{\uparrow,2j-1}c_{\uparrow,2j} + c_{\downarrow,2j-1}c_{\downarrow,2j}) \tag{5}$$

$$+\sin\theta(c_{\uparrow,2j-1}c_{\downarrow,2j+1} + c_{\downarrow,2j+2}c_{\uparrow,2j})]. \tag{6}$$

- One can check that the fermion parity of the ground state with a single texture $\theta = 0 \to 2\pi$ is (-1).

Let us compute the BdG Hamiltonian. Introduce the Nambu representation $\psi_{\sigma,j} = (a_{\sigma,j}, a_{\sigma,j}^\dagger)^T$. We have

$$\frac{i}{2}\sum_j c_{\sigma,2j-1}c_{\sigma,2j} = \frac{1}{2}\sum_j (a_{\sigma,j} + a_{\sigma,j}^\dagger)(a_{\sigma,j} - a_{\sigma,j}^\dagger) \tag{7}$$

$$= \frac{1}{2}\sum_j (a_{\sigma,j}^\dagger a_{\sigma,j} - a_{\sigma,j}a_{\sigma,j}^\dagger) \tag{8}$$

$$= \frac{1}{2}\sum_j \psi_{\sigma,j}^\dagger \tau_z \psi_{\sigma,j}. \tag{9}$$

Next,

$$\frac{i}{2}\sum_j (c_{\uparrow,2j-1}c_{\downarrow,2j-1} + c_{\downarrow,2j}c_{\uparrow,2j}) \tag{10}$$

$$= \frac{1}{2}\sum_j (a_{\uparrow,j}^\dagger, a_{\downarrow,j}^\dagger, a_{\uparrow,j}, a_{\downarrow,j}) \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{\uparrow,j} \\ a_{\downarrow,j} \\ a_{\uparrow,j}^\dagger \\ a_{\downarrow,j}^\dagger \end{pmatrix} \tag{11}$$

$$= \frac{1}{2}\sum_j \psi_j^\dagger(-\tau_x\sigma_y)\psi_j. \tag{12}$$

Similarly,

$$\frac{i}{2}\sum_j (c_{\uparrow,2j-1}c_{\downarrow,2j+1} + c_{\downarrow,2j+2}c_{\uparrow,2j}) \tag{13}$$

$$= \frac{1}{2}\sum_k (a^\dagger_{\uparrow,k}, a^\dagger_{\downarrow,k}, a_{\uparrow,-k}, a_{\downarrow,-k}) \begin{pmatrix} 0 & 0 & 0 & ie^{ik} \\ 0 & 0 & -ie^{-ik} & 0 \\ 0 & ie^{ik} & 0 & 0 \\ -ie^{-ik} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{\uparrow,k} \\ a_{\downarrow,k} \\ a^\dagger_{\uparrow,-k} \\ a^\dagger_{\downarrow,-k} \end{pmatrix} \tag{14}$$

$$= \frac{1}{2}\sum_k \psi^\dagger_k(-\cos k\,\tau_x\sigma_y - \sin k\,\tau_x\sigma_x)\psi_k. \tag{15}$$

Here the Fourier transform is

$$\psi_j = \sum_k \psi_k e^{ikj} = (a_k, a^\dagger_{-k}). \tag{16}$$

Therefore the BdG Hamiltonians are

$$\mathcal{H}_I(k,\theta) = \cos\theta\,\tau_z - \sin\theta\,\tau_x\sigma_y, \tag{17}$$

$$\mathcal{H}_{II}(k,\theta) = \cos\theta\,\tau_z - \sin\theta(\cos k\,\tau_x\sigma_y + \sin k\,\tau_x\sigma_x). \tag{18}$$

- Is the $\mathbb{Z}_2$ number defined on the (k,θ) space nontrivial?