The Čech–de Rham Complex

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Abstract

Following Chapter 8 of Bott–Tu, I summarize the Čech–de Rham complex.[1]

Let M be an oriented manifold and let $\mathcal{U} = \{U_\alpha\}_\alpha$ be an open cover, not necessarily a good cover. Write

$$U_{\alpha_0 \cdots \alpha_p} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}. \tag{1}$$

Also write $\Omega^* = \Omega^*(M)$. The Čech–de Rham complex is

$$C^*(\mathcal{U}, \Omega^*) = \bigoplus_{p,q \geq 0} K^{p,q} = \bigoplus_{p,q \geq 0} C^p(\mathcal{U}, \Omega^q). \tag{2}$$

A q-form-valued p-cochain $\omega \in C^p(\mathcal{U}, \Omega^q)$ is a collection

$$\omega = \{\omega_{\alpha_0 \cdots \alpha_p} \in \Omega^q(U_{\alpha_0 \cdots \alpha_p}) | \omega_{\alpha_0 \cdots \alpha_p} \text{ is skew symmetric in } \alpha_0, \dots, \alpha_p\}_{\alpha_0 \cdots \alpha_p}. \tag{3}$$

The exterior derivative

$$d : K^{p,q} \to K^{p,q+1} \tag{4}$$

is defined as usual. Define the cochain differential

$$\delta : K^{p,q} \to K^{p+1,q} \tag{5}$$

by

$$(\delta\omega)_{\alpha_0 \cdots \alpha_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \omega_{\alpha_0 \cdots \breve{\alpha}_j \cdots \alpha_{p+1}}. \tag{6}$$

Here $\breve{\alpha}_j$ means that α_j is omitted. Then $\delta^2 = 0$.

- The δ-cohomology vanishes.

Proof. Let ρ_α be a partition of unity, so

$$\sum_\alpha \rho_\alpha = 1, \quad \rho_\alpha \geq 0. \tag{7}$$

For $\omega \in K^{p,q}$, define $K\omega \in K^{p-1,q}$ by

$$(K\omega)_{\alpha_1 \cdots \alpha_p} = \sum_\alpha \rho_\alpha \omega_{\alpha\alpha_1 \cdots \alpha_p}. \tag{8}$$

[1] In Bott–Tu, the definition of the D-coboundary seems ambiguous, and the proof of injectivity of r^* in Proposition 8.8 seems incorrect to me.

Then

$$(\delta K\omega)_{\alpha_0\cdots\alpha_p} = \sum_{j=0}^{p}(-1)^j(K\omega)_{\alpha_0\cdots\check{\alpha}_j\cdots\alpha_p} \tag{9}$$

$$= \sum_{j=0}^{p}(-1)^j(K\omega)_{\alpha_0\cdots\check{\alpha}_j\cdots\alpha_p} \tag{10}$$

$$= \sum_{j=0}^{p}(-1)^j\sum_{\alpha}\rho_\alpha\omega_{\alpha\alpha_0\cdots\check{\alpha}_j\cdots\alpha_p}. \tag{11}$$

On the other hand,

$$(K\delta\omega)_{\alpha_0\cdots\alpha_p} = \sum_{\alpha}\rho_\alpha(\delta\omega)_{\alpha\alpha_0\cdots\alpha_p} \tag{12}$$

$$= \sum_{\alpha}\rho_\alpha\left(\omega_{\alpha_0\cdots\alpha_p} + \sum_{j=0}^{p}(-1)^{j+1}\omega_{\alpha\alpha_0\cdots\check{\alpha}_j\cdots\alpha_p}\right) \tag{13}$$

$$= \omega_{\alpha_0\cdots\alpha_p} + \sum_{\alpha}\rho_\alpha\sum_{j=0}^{p}(-1)^{j+1}\omega_{\alpha\alpha_0\cdots\check{\alpha}_j\cdots\alpha_p}. \tag{14}$$

Hence

$$\delta K + K\delta = 1. \tag{15}$$

The operator K is called a homotopy operator. Thus if $\delta\omega = 0$, then

$$\omega = \delta K\omega, \tag{16}$$

so ω is δ-exact. □

For the double complex $K^{p,q}$, define the differential D by

$$D = D' + D'' = \delta + (-1)^p d \quad \text{on } K^{p,q}. \tag{17}$$

Since $D' = \delta$ changes the p-degree by one, $D'D'' = -D''D'$. Therefore $D^2 = 0$. The D-cocycles, D-coboundaries, and D-cohomology are defined as usual:

$$Z_D^n = \ker[D : \bigoplus_{p+q=n} K^{p,q} \to \bigoplus_{p+q=n+1} K^{p,q}], \tag{18}$$

$$B_D^n = \operatorname{im}[D : \bigoplus_{p+q=n-1} K^{p,q} \to \bigoplus_{p+q=n} K^{p,q}], \tag{19}$$

$$H_D^*\{C^*(\mathcal{U},\Omega^*)\} = \bigoplus_{n\geq 0} Z_D^n/B_D^n. \tag{20}$$

For example, a 2-cocycle consists of the following data:

$$\omega = a + b + c, \quad a \in K^{0,2}, \quad b \in K^{1,1}, \quad c \in K^{2,0}, \tag{21}$$

$$da = 0, \tag{22}$$

$$\delta a - db = 0, \tag{23}$$

$$\delta b + dc = 0, \tag{24}$$

$$\delta c = 0. \tag{25}$$

For example, a 4-coboundary consists of the following data:[2]

$$\omega = b_0 + b_1 + b_2 + b_3 + b_4, \quad b_j \in K^{j,4-j}, \tag{26}$$
$$b_0 = da_0, \tag{27}$$
$$b_1 = \delta a_0 - da_1, \tag{28}$$
$$b_2 = \delta a_1 + da_2, \tag{29}$$
$$b_3 = \delta a_2 - da_3, \tag{30}$$
$$b_4 = \delta a_4, \tag{31}$$
$$a_j \in K^{j,3-j}. \tag{32}$$

$$H^*_{\mathrm{dR}}(M) \cong H_D\{C^*(\mathcal{U}, \Omega^*)\}. \tag{33}$$

Proof. Define the restriction map

$$r : \Omega^*(M) \to C^0(\mathcal{U}, \Omega^*), \tag{34}$$
$$\omega \mapsto (r\omega)_\alpha := \omega \quad \text{on } U_\alpha. \tag{35}$$

First we show that r induces a map to $H_D\{C^*(\mathcal{U}, \Omega^*)\}$, namely that if ω is d-exact then $r\omega$ is D-exact. Let $\omega = d\eta$. Since

$$(\delta r\omega)_{\alpha\beta} = (r\omega)_\beta - (r\omega)_\alpha = 0 \text{ on } U_{\alpha\beta}, \tag{36}$$

we have

$$\delta r = 0. \tag{37}$$

Also

$$(rd\omega)_\alpha = d\omega \text{ on } U_\alpha, \tag{38}$$
$$(dr\omega)_\alpha = d(r\omega)_\alpha = d(\omega \text{ on } U_\alpha) = d\omega \text{ on } U_\alpha, \tag{39}$$

so $rd = dr$. Thus, when $\omega = d\eta$, $Dr\omega = (\delta + d)r\omega = rd\omega = 0$. Therefore restriction induces

$$r^* : H^*_{\mathrm{dR}}(M) \to H_D\{C^*(\mathcal{U}, \Omega^*)\}. \tag{40}$$

We prove that this is an isomorphism.

First we prove surjectivity of r^*. Let $D\omega = 0$. As an example, suppose that ω is a 2-cocycle:

$$\omega = a + b + c, \quad a \in K^{0,2}, \quad b \in K^{1,1}, \quad c \in K^{2,0}, \tag{41}$$
$$da = 0, \tag{42}$$
$$\delta a - db = 0, \tag{43}$$
$$\delta b + dc = 0, \tag{44}$$
$$\delta c = 0. \tag{45}$$

Since $\delta c = 0$, there exists $X \in K^{1,0}$ with $c = \delta X$. Then

$$\omega = a + b + \delta X = a + (b + dX) + DX, \quad b + dX \in K^{1,1}. \tag{46}$$

Since $a + (b + dX)$ is also a D-cocycle, $\delta(b + dX) = 0$. Indeed,

$$\delta(b + dX) = \delta b + \delta dX = \delta b + dc = 0. \tag{47}$$

[2] Looking at the figure on p. 96 of Bott–Tu, it may seem that $da_0 = 0$ and $\delta a_4 = 0$ are also included in the definition, but they should not be necessary.

Therefore there exists $Y \in K^{0,1}$ such that

$$b + dX = \delta Y. \tag{48}$$

Thus

$$\omega = a + \delta Y + DX = (a - dY) + DY + DX, \quad a - dY \in K^{0,2}. \tag{49}$$

Since $a - dY$ is also a D-cocycle, $\delta(a - dY) = 0$. Indeed,

$$\delta(a - dY) = \delta a - d(b + dX) = \delta a - db = 0. \tag{50}$$

Therefore $a - dY$ is a global 2-form. Moreover,

$$d(a - dY) = 0, \tag{51}$$

so $a - dY$ is d-closed. Hence $r^*([a - dY]_d) = [\omega]_D$. The same argument works for a general p-form.

Next we prove injectivity of r^*. Let $\omega = d\eta$ be d-exact. It suffices to show that $r^*([\omega]_d) = 0$, namely that $r\omega$ is D-exact. As already shown, $rd = dr = Dr$, so

$$[r\omega]_D = [rd\eta]_D = [Dr\eta]_D = 0. \tag{52}$$

This proves injectivity. □

References

[1] Raoul Bott and Loring W. Tu, *Differential Forms in Algebraic Topology.*