Reading Cheeger–Simons

Ken Shiozaki

May 31, 2026

Abstract

I follow [1]. For details such as proofs, [2] is more thorough.

1 Differential Characters

1.1 What is a proper subring of $\mathbb{R}$?

Let Λ be a proper subring of $\mathbb{R}$. That is, $\Lambda \subset \mathbb{R}$, $\Lambda \neq \mathbb{R}$, and Λ is closed under multiplication and addition.

Proposition 1.1. *If an additive group $\Lambda \subset \mathbb{R}$ contains a nontrivial open interval (a, b), then $\Lambda = \mathbb{R}$.*

(Proof) Let $a < b$. Notice that $\frac{a+b}{2} \in \Lambda$. Since Λ is an additive group, $(\frac{a-b}{2}, \frac{b-a}{2}) = \{x - \frac{a+b}{2} \in \mathbb{R} | x \in (a, b)\} \subset \Lambda$. For any real number $r \in \mathbb{R}$, there exists an integer n such that $\frac{1}{n} r \in (\frac{a-b}{2}, \frac{b-a}{2})$. □

Thus Λ contains no continuous interval. The group Λ is either discrete or dense. Here, a subset $S \subset \mathbb{R}$ is called discrete if, for every $x \in S$, there exists $\epsilon > 0$ such that the ϵ-neighborhood $(x - \epsilon, x + \epsilon)$ contains no point of S other than x. A subset $S \subset \mathbb{R}$ is called dense if, for every $x \in S$ and every $\epsilon > 0$, there exists $y \in S$ such that $|x - y| < \epsilon$, namely points of S exist arbitrarily close to x. By definition, a subset $S \in \mathbb{R}$ cannot be both discrete and dense.

Proposition 1.2. *Any additive subgroup $\Lambda \in \mathbb{R}$ is either discrete or dense.*

(Proof) Assume that Λ is not discrete. That is, there exists $x \in \Lambda$ such that points of Λ exist arbitrarily close to x. (For every $\epsilon > 0$, $(x - \epsilon, x + \epsilon) \cap \Lambda \neq \emptyset$.) Since Λ is an additive group, we may take $x = 0$. Thus there are points of Λ arbitrarily close to 0. (For every $\epsilon > 0$, there exists $x \in \Lambda$ such that $|x| < \epsilon$.) Using integer multiples of such an x, we can approximate any real number $r \in \mathbb{R}$ to arbitrary precision. Namely, for every $\epsilon > 0$, there exist $x \in \Lambda$ and $n \in \mathbb{Z}$ such that $|r - nx| < \epsilon$ and $nx \in \Lambda$. Hence Λ is dense. □

Proposition 1.3. *Any discrete additive subgroup $\Lambda \in \mathbb{R}$ is of the form $\Lambda = \alpha\mathbb{Z}$.*

(Proof) Since Λ is discrete, for some $\epsilon > 0$ one has $\alpha = \min_{x,y \in \Lambda, x \neq y} |x - y| > \epsilon$. By additivity, $\alpha \in \Lambda$ and $\alpha\mathbb{Z} \subset \Lambda$. If there existed $x \in \Lambda$ with $x \notin \alpha\mathbb{Z}$, this would contradict the definition of α. □

On the other hand, a dense Λ need not be the set $\mathbb{Q}$ of all rational numbers. For example, $\mathbb{Z} + \sqrt{2}\mathbb{Z} = \{n + m\sqrt{2} | n, m \in \mathbb{Z}\}$ is clearly an additive subgroup, but it contains irrational numbers and is not of the form $\alpha\mathbb{Z}$, so it is not discrete and hence is dense. Moreover, since $(n + \sqrt{2}m)(n' + \sqrt{2}m') = (nn' + 2mm') + \sqrt{2}(nm' + mn') \in \mathbb{Z} + \sqrt{2}\mathbb{Z}$, the set $\mathbb{Z} + \sqrt{2}\mathbb{Z}$ is a subring.

1.2 Preparations

1.2.1 Notation p, r, ι

Consider the short exact sequence

$$0 \to \Lambda \xrightarrow{r} \mathbb{R} \xrightarrow{p} \mathbb{R}/\Lambda \to 0. \tag{1.1}$$

The projection $p : \mathbb{R} \to \mathbb{R}/\Lambda$ and the embedding $r : \Lambda \to \mathbb{R}$ induce homomorphisms of cochains

$$p_* : C^k(M, \mathbb{R}) \to C^k(M, \mathbb{R}/\Lambda), \quad (p_* f)(\sigma) := p(f(\sigma)), \tag{1.2}$$

$$r_* : C^k(M, \Lambda) \to C^k(M, \mathbb{R}), \quad (r_* f)(\sigma) := r(f(\sigma)). \tag{1.3}$$

Both are chain maps:

$$((p_*\delta_{\mathbb{R}} - \delta_{\mathbb{R}/\Lambda} p_*)f)(\sigma) = p(\delta_{\mathbb{R}} f(\sigma)) - (p_* f)(\partial\sigma) = p(f(\partial\sigma)) - p(f(\partial\sigma)) = 0, \tag{1.4}$$

$$((r_*\delta_{\Lambda} - \delta_{\mathbb{R}} r_*)f)(\sigma) = r(\delta_{\Lambda} f(\sigma)) - (r_* f)(\partial\sigma) = r(f(\partial\sigma)) - r(f(\partial\sigma)) = 0. \tag{1.5}$$

Define the de Rham homomorphism by

$$\iota : \Omega^k(M) \to C^k(M, \mathbb{R}), \quad \omega \mapsto \iota(\omega) = \left(\sigma \mapsto \int_\sigma \omega\right). \tag{1.6}$$

The map ι is also a chain map:

$$((\iota d - \delta_{\mathbb{R}}\iota)\omega)(\sigma) = \int_\sigma d\omega - \iota\omega(\partial\sigma) = \int_\sigma d\omega - \int_{\partial\sigma} \omega = 0. \tag{1.7}$$

Below, when it is clear from context, all of $\delta_{\mathbb{R}}, \delta_\Lambda, \delta_{\mathbb{R}/\Lambda}$ will simply be denoted by δ.

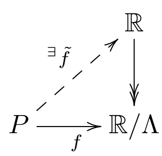
1.2.2 Projective objects

In general, for a free abelian group P, the functor $\mathrm{Hom}(P, -)$ is exact, and hence the following is a short exact sequence:

$$0 \to \mathrm{Hom}(P, \Lambda) \xrightarrow{r_*} \mathrm{Hom}(P, \mathbb{R}) \xrightarrow{p_*} \mathrm{Hom}(P, \mathbb{R}/\Lambda) \to 0. \tag{1.8}$$

Therefore,

- if $r_* f = 0$ for $f \in \mathrm{Hom}(P, \Lambda)$, then $f = 0$;
- if $p_* g = 0$ for $g \in \mathrm{Hom}(P, \mathbb{R})$, then there exists $f \in \mathrm{Hom}(P, \Lambda)$ such that $g = r_* f$;
- (the projective property of free abelian groups) for any $f \in \mathrm{Hom}(P, \mathbb{R}/\Lambda)$ there exists $\tilde{f} \in \mathrm{Hom}(P, \mathbb{R})$ such that $f = p_*\tilde{f}$:

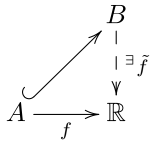


(1.9)

Below we use these facts with $P = C_k(M), Z_k(M)$. (The group $C_k(M)$ is the free abelian group with basis the set of cubes $\sigma : I^k \to M$, and in general any subgroup of a free abelian group is a free abelian group.)

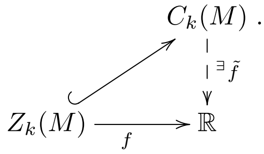
1.2.3 Injective objects

As a general fact, $\mathbb{R}$ is an injective object as a $\mathbb{Z}$-module. Thus, for an injection $A \to B$ and a homomorphism $A \to \mathbb{R}$, there exists $\tilde{f}$ making the following diagram commute:



(1.10)

Below we use it in the form



(1.11)

1.2.4 On $f = \delta g$ when $f|_{Z_k} = 0$

- Let the following be an exact sequence of abelian groups:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C. \tag{1.12}$$

The functor $\mathrm{Hom}(M, -)$ is a left exact covariant functor. Namely, the following is exact:

$$0 \to \mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C). \tag{1.13}$$

(Proof) For $\alpha : M \to A$, suppose that $f_*\alpha = f \circ \alpha = 0$. Then for every $m \in M$, $0 = f_*\alpha(m) = f(\alpha(m))$. Since f is injective, $\alpha(m) = 0$, and hence $\alpha = 0$. Let $\alpha : M \to A$. Since $g \circ f = 0$, for $m \in M$ we have $g_* f_* \alpha(m) = g \circ f(\alpha(m)) = 0$, and hence $g_* f_* = 0$. Let $\beta : M \to B$ satisfy $g_*\beta = 0$. For $m \in M$, $g_*\beta(m) = g(\beta(m)) = 0$. Since $\ker g = \mathrm{im}\, f$, there exists $a_m \in A$ such that $\beta(m) = a_m$. The homomorphism property of the correspondence $m \mapsto a_m$ follows from the linearity of β: $a_m + a_{m'} = \beta(m) + \beta(m') = \beta(m + m') = a_{m+m'}$. □

- Let the following be an exact sequence of abelian groups:

$$M \xrightarrow{f} N \xrightarrow{g} L \to 0. \tag{1.14}$$

The functor $\mathrm{Hom}(-, A)$ is a left exact contravariant functor. Namely, the following is exact:

$$\mathrm{Hom}(M, A) \xleftarrow{f^*} \mathrm{Hom}(N, A) \xleftarrow{g^*} \mathrm{Hom}(L, A) \leftarrow 0. \tag{1.15}$$

Wiebel's textbook only says, "since $\mathrm{Hom}_{\mathcal{A}}(A, M) = \mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(M, A)$," but I check the proof directly.

(Proof) For $\gamma : L \to A$, suppose that $g^*\gamma = 0$. Then for every $n \in N$, $g^*\gamma(n) = \gamma(g(n)) = 0$. Since g is surjective, $\gamma = 0$. For $\gamma : L \to A$, $f^* g^* \gamma(m) = \gamma(f \circ g(m)) = \gamma(0) = 0$, so $f^* g^* = 0$. Let $\beta \in \ker f^*$. This means that, for every $m \in M$, $f^*\beta(m) = \beta(f(m)) = 0$. In other words, β is zero on $\mathrm{im}\, f = \ker g$. Since g is surjective, for any $l \in L$ there exists $n_l \in N$ such that $l = g(n_l)$. Define $\gamma : L \to A$ by

$$\gamma(l) = \beta(n_l). \tag{1.16}$$

This is well-defined: if $g(n'_l) = g(n_l)$, then $n_l - n'_l \in \ker g = \mathrm{im}\, f$, and hence $\beta(n_l) - \beta(n'_l) = \beta(n_l - n'_l) = 0$. The homomorphism property follows from $g(n_l + n_{l'}) = g(n_l) + g(n_{l'}) = l + l'$, so one may take $n_{l+l'} = n_l + n_{l'}$. □

Apply the latter statement as follows. By the definition of Z_k, the following is an exact sequence:

$$0 \to Z_k(M) \xrightarrow{i} C_k(M) \xrightarrow{\partial} B_{k-1}(M) \to 0. \tag{1.17}$$

By left exactness of the functor $\mathrm{Hom}(-, A)$ for an abelian group A, the following is exact:

$$\mathrm{Hom}(Z_k(M), A) \xleftarrow{i^*} \mathrm{Hom}(C_k(M), A) \xleftarrow{\partial^*(=\delta)} \mathrm{Hom}(B_{k-1}(M), A) \leftarrow 0. \tag{1.18}$$

Here i^* is the restriction map. Applying the above, $\ker i^* = \mathrm{im}\,\delta$. Thus if a cochain $f : C^k(M, A)$ satisfies $i^* f = f|_{Z_k(M)} = 0$ on its restriction to $Z_k(M)$, then there exists $g : B_{k-1} \to A$ such that

$$f = \delta g = g \circ \partial. \tag{1.19}$$

In particular, when $A = \mathbb{R}$, the injectivity of $\mathbb{R}$ allows one to extend the domain to $C_{k-1}(M)$. In summary, we have shown:

- if the restriction of $f \in C^k(M, \mathbb{R})$ to $Z_k(M)$ is zero, $f|_{Z_k(M)} = 0$, then there exists $g \in C^{k-1}(M, \mathbb{R})$ such that $f = \delta g$.

We will also use the following form:

$$\mathrm{Hom}(Z_k(M), \mathbb{R}/\Lambda) \xleftarrow{i^*} \mathrm{Hom}(C_k(M), \mathbb{R}/\Lambda) \xleftarrow{\partial^*(=\delta)} \mathrm{Hom}(B_{k-1}(M), \mathbb{R}/\Lambda) \leftarrow 0. \tag{1.20}$$

1.3 $p_*\iota$ is injective

If $p_*\iota\omega = 0$ for a k-form $\omega \in \Omega^k(M)$, then, since $\ker p_* = \mathrm{im}\, r_*$, there exists $c \in C^k(M, \Lambda)$ such that

$$\iota\omega = r_* c. \tag{1.21}$$

This is equivalent to

$$\int_\sigma \omega = c(\sigma) \in \Lambda \tag{1.22}$$

for every singular cube $\sigma : I^k \to M$. Since the proper subring Λ contains no continuous interval (a, b), if ω were a nonzero k-form this would be a contradiction. Therefore

$$\omega = 0 \tag{1.23}$$

holds, and hence

$$p_*\iota : \Omega^k(M) \to C^k(M, \mathbb{R}/\Lambda) \tag{1.24}$$

is injective. The injectivity of r_* also implies

$$\iota\omega = r_* c \quad \Rightarrow \quad c = 0. \tag{1.25}$$

1.3.1 The Bockstein homomorphism

Let

$$B : H^k(M, \mathbb{R}/\Lambda) \to H^{k+1}(M, \Lambda) \tag{1.26}$$

be the Bockstein homomorphism. Explicitly, for $a \in Z^k(M, \mathbb{R}/\Lambda)$, choose a lift $\tilde{a} \in C^k(M, \mathbb{R})$ satisfying $p_*\tilde{a} = a$. Since $p_*\delta\tilde{a} = \delta p_*\tilde{a} = \delta a = 0$, there exists $c \in C^k(M, \Lambda)$ such that $\delta\tilde{a} = r_* c$. Then define

$$B([a]) := [c]. \tag{1.27}$$

Notice that c is a cocycle, since $r_*\delta c = 0$. The change induced by replacing $\tilde{a} \mapsto \tilde{a} + r_* b$, $b \in C^k(M, \Lambda)$, is a coboundary, since $\delta r_* b = r_*\delta b$. The change induced by replacing $a \mapsto a + \delta d = a + d \circ \partial$ is as follows. If $\tilde{d} \in C^{k-1}(M, \mathbb{R})$ denotes an extension of $d : B_{k-1} \to \mathbb{R}/\Lambda$ to $C_{k-1} \to \mathbb{R}$, then $p_*\delta\tilde{d} = \delta d$. Thus $\delta\tilde{a} \mapsto \delta\tilde{a} + \delta^2\tilde{d} = \delta\tilde{a}$, so the definition is independent of the representative a.

1.4 Memo

An abelian group G is divisible if, for any $g \in G$ and any $n \in \mathbb{N}$, there exists $h \in G$ such that $nh = g$. In other words, every element can be divided by any nonzero integer.

An abelian group G is injective if, for any injective homomorphism $f : A \to B$ and any homomorphism $g : A \to G$, there exists a homomorphism $h : B \to G$ such that $g = h \circ f$.

For abelian groups, divisibility and injectivity are equivalent.

The group $\mathbb{R}$ is injective as a $\mathbb{Z}$-module.

Free $\mathbb{Z}$-modules are projective.

1.5 Structure theorem for differential characters

Definition 1.4.

$$\hat{H}^k(M, \mathbb{R}/\Lambda) := \{f \in \mathrm{Hom}(Z_k(M), \mathbb{R}/\Lambda) | {}^{\exists}\omega_f \in \Omega^{k+1}(M) \text{ s.t. } f \circ \partial = p_* \iota \omega_f\}. \tag{1.28}$$

Here, note that

$$C_{k+1}(M) \xrightarrow{\partial} B_k(M) \subset Z_k(M) \xrightarrow{f} \mathbb{R}/\Lambda. \tag{1.29}$$

I go through Theorem 1.1 of [1] step by step.

Proposition 1.5. *For $f \in \mathrm{Hom}(Z_k(M), \mathbb{R}/\Lambda)$, there exists $T_f \in C^k(M, \mathbb{R})$ such that*

$$p_* T_f|_{Z_k(M)} = f. \tag{1.30}$$

(Proof) First note that any homomorphism $f : Z_k(M) \to \mathbb{R}/\Lambda$ can be lifted to a homomorphism $\tilde{f} : Z_k(M) \to \mathbb{R}$. Indeed, $Z_k(M) \subset C_k(M)$, and $C_k(M)$ is the free abelian group generated by nondegenerate cubes $\sigma : I^k \to M$. Since a subgroup of a free abelian group is again free, $Z_k(M)$ also has a basis, namely it is the free abelian group generated by some set $\mathcal{S}$. For each $s \in \mathcal{S}$, choose one lift $\mathbb{R}/\Lambda \ni f(s) \mapsto \widetilde{f(s)} \in \mathbb{R}$, and define

$$\tilde{f}(s) := \widetilde{f(s)}. \tag{1.31}$$

Then extend linearly to a general $\sigma \in Z_k(M)$. Thus we obtain $\tilde{f}$ satisfying

$$p_* \tilde{f} = f \tag{1.32}$$

[1]:

$$\begin{array}{ccc} & & \mathbb{R} \\ & {}^{\exists}\tilde{f} \nearrow & \downarrow \\ Z_k(M) & \xrightarrow[f]{} & \mathbb{R}/\Lambda \end{array} \quad . \tag{1.33}$$

Since $\mathbb{R}$ is divisible, it is injective, and for the injection $Z_k(M) \to C_k(M)$ the domain of the homomor-

[1] One can simply apply the general fact (1.9), but I wrote down a constructive proof.

phism $\tilde{f} : Z_k(M) \to \mathbb{R}$ can be extended:

$$\begin{array}{ccc} & & C_k(M) \\ & \nearrow & \Big\downarrow {\scriptstyle \exists T_f \in C^k(M,\mathbb{R})} \\ Z_k(M) & \xrightarrow[\tilde{f}]{} & \mathbb{R} \end{array} . \tag{1.34}$$

Thus we obtain T_f such that

$$T_f|_{Z_k(M)} = \tilde{f}. \tag{1.35}$$

Notice that the restriction map commutes with p_*, so $p_*(T_f|_{Z_k(M)}) = (p_*T_f)|_{Z_k(M)}$, and hence the notation $p_*T_f|_{Z_k(M)}$ is justified. □

Below, T_f may be called a representation of f.

Proposition 1.6. *For $f \in \hat{H}^k(M, \mathbb{R}/\Lambda)$, the form $\omega_f \in \Omega^{k+1}(M)$ satisfying $f \circ \partial = p_*\iota\omega_f$ is closed and has periods in Λ. Namely,*

$$d\omega_f = 0, \tag{1.36}$$

$$\int_\sigma \omega_f \in \Lambda \text{ for all } \sigma \in Z_{k+1}(M). \tag{1.37}$$

(Proof) Choose $T_f \in C^k(M,\mathbb{R})$ such that $f = p_*T_f|_{Z_k(M)}$. Since $\partial C_{k+1}(M) = B_k(M) \subset Z_k(M)$,

$$p_*\delta T_f = \delta p_*T_f = p_*T_f \circ \partial = f \circ \partial = p_*\iota\omega_f. \tag{1.38}$$

Thus there exists $c_f \in C^{k+1}(M, \Lambda)$ such that

$$\delta T_f = \iota\omega_f - r_*c_f. \tag{1.39}$$

It follows that

$$0 = \delta^2 T_f = \delta\iota\omega_f - \delta r_*c_f = \iota d\omega_f - r_*\delta c_f, \tag{1.40}$$

or

$$\iota d\omega_f = r_*\delta c_f. \tag{1.41}$$

By the discussion in Sec. 1.3,

$$d\omega_f = 0, \quad \delta c_f = 0. \tag{1.42}$$

The period condition follows because, for any $\sigma \in Z_{k+1}(M)$,

$$\int_\sigma \omega_f = (\iota\omega_f)(\sigma) = (\delta T_f + r_*c_f)(\sigma) = (r_*c_f)(\sigma) \in \Lambda. \tag{1.43}$$

□

Write the closed forms with periods in Λ as

$$\Omega^k_{\mathrm{cl}}(M)_\Lambda := \{\omega \in \Omega^k_{\mathrm{cl}}(M) | \int_\sigma \omega \in \Lambda \text{ for all } \sigma \in Z_k(M)\} \tag{1.44}$$

$$= \{\omega \in \Omega^k(M) | d\omega = 0, \int_\sigma \omega \in \Lambda \text{ for all } \sigma \in Z_k(M)\}. \tag{1.45}$$

The definition of a differential character can be rewritten as

$$\hat{H}^k(M, \mathbb{R}/\Lambda) = \{f \in \mathrm{Hom}(Z_k(M), \mathbb{R}/\Lambda) | {}^\exists \omega_f \in \Omega^{k+1}_{\mathrm{cl}}(M)_\Lambda \text{ s.t. } f \circ \partial = p_* \iota \omega_f\}. \tag{1.46}$$

Proposition 1.7. *The form ω_f is unique. Equivalently, $p_*\iota : \Omega^{k+1}(M) \to C^{k+1}(M, \mathbb{R}/\Lambda)$ is injective.*

(Proof) If ω'_f also satisfies $f \circ \partial = p_*\iota\omega'_f$, then $p_*\iota(\omega_f - \omega'_f) = 0$, and $p_*\iota$ is injective. □

Write the cohomology class determined by c_f as

$$u_f := [c_f] \in H^{k+1}(M, \Lambda). \tag{1.47}$$

Notice that

$$[\iota\omega_f] = [r_* c_f] = r_*[c_f] = r_* u_f. \tag{1.48}$$

Proposition 1.8. *The class u_f is unique.*

(Proof) The cochain c_f is determined by the relation

$$r_* c_f = \delta T_f - \iota\omega_f. \tag{1.49}$$

The form ω_f is unique, but T_f has ambiguity. Let $T'_f \in C^k(M, \mathbb{R})$ be another choice satisfying $p_* T'_f|_{Z_k(M)} = f$, and define

$$r_* c'_f = \delta T'_f - \iota\omega_f. \tag{1.50}$$

Then

$$p_*(T_f - T'_f)|_{Z_k(M)} = 0. \tag{1.51}$$

Thus there exists $d_0 : Z_k(M) \to \Lambda$ such that

$$(T_f - T'_f)|_{Z_k(M)} = r_* d_0. \tag{1.52}$$

The homomorphism $r_* d_0 : Z_k \to \mathbb{R}$ can be extended by the injectivity of $\mathbb{R}$ as a $\mathbb{Z}$-module, so there exists $d_1 \in C^k(M, \mathbb{R})$ such that

$$r_* d_0 = d_1|_{Z_k(M)}. \tag{1.53}$$

Therefore,

$$(T_f - T'_f - d_1)|_{Z_k(M)} = 0. \tag{1.54}$$

By the remark in Sec. 1.2.4, there exists $s \in C^{k-1}(M, \mathbb{R})$ such that

$$T_f - T'_f - d_1 = \delta s. \tag{1.55}$$

Thus

$$r_*(c'_f - c_f) = \delta T'_f - \delta T_f = -\delta d_1 \tag{1.56}$$

is a coboundary, and hence, by the injectivity of r_*,

$$[c'_f] = [c_f]. \quad \square \tag{1.57}$$

We have shown that ω_f and u_f are uniquely determined by $f \in \hat{H}^k(M, \mathbb{R}/\Lambda)$. Let us also record the ambiguity of the representation T_f.

Proposition 1.9. *The ambiguity of a representation T_f of f is*

$$T_f \mapsto T_f + r_* d + \delta s, \quad d \in C^k(M, \Lambda), \quad s \in C^{k-1}(M, \mathbb{R}). \tag{1.58}$$

(Proof) Trace the above argument backward. The difference $T = T'_f - T_f$ satisfies

$$p_* T|_{Z_k(M)} = 0. \tag{1.59}$$

By the exact sequence (1.20),

$$p_* T = \delta b, \quad b \in C^{k-1}(M, \mathbb{R}/\Lambda). \tag{1.60}$$

Choose a lift $b \mapsto s \in C^{k-1}(M, \mathbb{R})$ with $p_* s = b$. Then

$$p_* T = \delta p_* s = p_* \delta s, \quad s \in C^{k-1}(M, \mathbb{R}). \tag{1.61}$$

Thus $p_*(T - \delta s) = 0$, and hence there exists $d \in C^k(M, \Lambda)$ such that

$$T - \delta s = r_* d. \quad \square \tag{1.62}$$

Now define

$$\delta_1 : \hat{H}^k(M, \mathbb{R}/\Lambda) \to \Omega^{k+1}_{\mathrm{cl}}(M)_\Lambda, \quad \delta_1(f) := \omega_f, \tag{1.63}$$
$$\delta_2 : \hat{H}^k(M, \mathbb{R}/\Lambda) \to H^{k+1}(M, \Lambda), \quad \delta_2(f) := u_f = [c_f]. \tag{1.64}$$

Proposition 1.10. *A form $\omega \in \Omega^{k+1}_{\mathrm{cl}}(M)_\Lambda$ determines $u \in H^{k+1}(M, \Lambda)$.*

(Proof) Let $\omega \in \Omega^{k+1}_{\mathrm{cl}}(M)_\Lambda$. By the period condition, $p_* \iota\omega|_{Z_k(M)} = 0$. In the exact sequence (1.20), $\iota\omega \in \ker i = \operatorname{im} \delta$, so

$$^\exists T : B_k(M) \to \mathbb{R}/\Lambda \quad \text{s.t.} \quad p_* \iota\omega = \delta T. \tag{1.65}$$

Since $B_k(M)$ is a free $\mathbb{Z}$-module, there exists $\tilde{T} : B_k(M) \to \mathbb{R}$ such that $p_* \tilde{T} = T$. Hence

$$p_*(\iota\omega - \delta\tilde{T}) = 0. \tag{1.66}$$

Therefore there exists $c \in C^{k+1}(M, \Lambda)$ such that

$$\iota\omega - \delta\tilde{T} = r_* c. \tag{1.67}$$

Since $r_* \delta c = 0$ and r_* is injective, the class $u = [c]$ is defined. If T' is another choice and $r_* c' = \iota\omega - \delta T'$, then $r_*(c - c') = -\delta(T - T')$, so u is unique. $\square$

Proposition 1.11. *An element $u \in H^{k+1}(M, \Lambda)$ determines $[\omega] \in H^{k+1}_{\mathrm{dR}}(M)$ with $\omega \in \Omega^{k+1}_{\mathrm{cl}}(M)_\Lambda$.*

(Proof) For $u \in H^{k+1}(M, \Lambda)$, consider $r_* u \in H^{k+1}(M, \mathbb{R})$. By the isomorphism $\iota^{-1} : H^{k+1}(M, \mathbb{R}) \cong$

$H^{k+1}_{\mathrm{dR}}(M)$, there exists $[\omega] \in H^{k+1}_{\mathrm{dR}}(M)$ such that $\iota^{-1}(r_* u) = [\omega]$. If $u = [c]$, then $r_* c = \iota\omega + \delta T$, and therefore

$$\int_{z \in Z_{k+1}(M)} \omega = (r_* c - \delta T)(z) = c(z) \in \Lambda. \tag{1.68}$$

Thus ω has periods in Λ. □

Proposition 1.12. *The maps δ_1, δ_2 are surjective.*

(Proof) By the preceding two propositions, if either $\omega \in \Omega^{k+1}_{\mathrm{cl}}(M)_\Lambda$ or $u \in H^{k+1}(M, \Lambda)$ is given, a pair ω, u satisfying $[\iota\omega] = r_* u$ is determined. Let $u = [c]$, $c \in C^{k+1}(M, \Lambda)$. Then

$$\delta T = \iota\omega - r_* c, \quad T \in C^k(M, \mathbb{R}) \tag{1.69}$$

is determined. Now set

$$f := p_* T|_{Z_k(M)}. \tag{1.70}$$

Then

$$f(\partial\sigma \in B_k(M) \subset Z_k(M)) = p_* T(\partial\sigma) = p_* \delta T(\sigma) = p_*(\iota\omega - r_* c)(\sigma) = p_* \iota\omega(\sigma), \tag{1.71}$$

so $f \circ \partial = p_* \iota\omega$, and $f \in \hat{H}^k(M, \mathbb{R}/\Lambda)$.

To show independence of the replacement of c, first show that f is independent of the choice of T. Under $T \mapsto T + \delta S$,

$$f(z \in Z_k(M)) = p_* T(z) \mapsto p_* T(z) + p_* \delta S(z) = p_* T(z) + p_* S(\partial z) = p_* T(z), \tag{1.72}$$

so f depends only on the class $[T] \in H^k(M, \mathbb{R})$. Moreover, replacing $c \mapsto c + \delta b$ is precisely $T \mapsto T + \delta r_* b$, so f is determined only by $u = [c] \in H^{k+1}(M, \Lambda)$. □

The proof also gives the following.

Corollary 1.13. *A triple*

$$(T, c, \omega) \in C^k(M, \mathbb{R}) \times C^{k+1}(M, \Lambda) \times \Omega^{k+1}(M) \tag{1.73}$$

satisfying

$$\delta T = \iota\omega - r_* c \tag{1.74}$$

defines a differential character $f := p_* T|_{Z_k(M)} \in \hat{H}^k(M, \mathbb{R}/\Lambda)$. The character f depends only on $[T] \in H^k(M, \mathbb{R})$ and $[c] \in H^{k+1}(M, \Lambda)$.

Although we wrote $\omega \in \Omega^{k+1}(M)$ here, the relation $\delta T = \iota\omega - r_* c$ implies $\omega \in \Omega^{k+1}_{\mathrm{cl}}(M)_\Lambda$, as shown above.

For $v \in H^k(M, \mathbb{R}/\Lambda)$, choose $s \in Z^k(M, \mathbb{R}/\Lambda)$ with $v = [s]$ and set

$$f = s|_{Z_k(M)}. \tag{1.75}$$

Replacing s by $s \mapsto s + \delta t$ does not change f, since $\delta t(z \in Z_k(M)) = 0$. Also $f(\partial\sigma) = 0$, so f is a differential character with $\delta_1(f) = 0$. Write this as

$$H^k(M, \mathbb{R}/\Lambda) \xrightarrow{i_1} \hat{H}^k(M, \mathbb{R}/\Lambda), \quad v = [s] \mapsto i_1([s]) = s|_{Z_k(M)}. \tag{1.76}$$

Clearly i_1 is injective. If we take an extension $p_*\tilde{s} = s$, $\tilde{s} \in C^k(M, \mathbb{R})$, to $\mathbb{R}$-valued cochains, then a representation of $f = i_1([s])$ is $T_f = \tilde{s}$. Then the cohomology class $-u_f = -[c_f]$ of the cochain $c \in C^{k+1}(M, \Lambda)$ determined by $\delta\tilde{s} = -r_* c_f$ is exactly the Bockstein homomorphism of $[s]$:

$$u_f = -B([s]). \tag{1.77}$$

Thus $f = i_1([s])$ is a differential character satisfying $\delta_1(f) = 0$ and $\delta_2(f) = -B([s])$. We have shown:

Proposition 1.14.

$$\delta_2 \circ i_1 = -B. \tag{1.78}$$

Proposition 1.15. *There is an exact sequence*

$$0 \to H^k(M, \mathbb{R}/\Lambda) \xrightarrow{i_1} \hat{H}^k(M, \mathbb{R}/\Lambda) \xrightarrow{\delta_1} \Omega^{k+1}_{\mathrm{cl}}(M)_\Lambda \to 0. \tag{1.79}$$

(Proof) The injectivity of i_1, the surjectivity of δ_1, and $\delta_1 \circ i_1 = 0$ have already been shown. Let $f \in \ker \delta_1$, namely $\omega_f = 0$. Choose a representation T_f such that $f = p_* T_f|_{Z_k(M)}$. Then $\delta T_f = -r_* c_f$. Thus

$$\delta T_f(\sigma) = T_f(\partial\sigma) = -r_* c_f(\sigma) \in \Lambda, \tag{1.80}$$

so $\delta p_* T_f = p_* \delta T_f = 0$, and a cocycle $p_* T_f \in Z^k(M, \mathbb{R}/\Lambda)$ is defined. Under a replacement $T_f \mapsto T_f + \delta S$, the change $p_*\delta S = \delta p_* S$ is a coboundary, so f determines a cohomology class $[p_* T_f] \in H^k(M, \mathbb{R}/\Lambda)$. Moreover $i_1([p_* T_f]) = p_* T_f|_{Z_k(M)}$. □

For a k-form $\alpha \in \Omega^k(M)$, define

$$f_\alpha = p_* \iota\alpha|_{Z_k(M)}, \tag{1.81}$$

that is,

$$f_\alpha(z \in Z_k(M)) = p\left(\int_z \alpha\right). \tag{1.82}$$

Then

$$f_\alpha(\partial\sigma) = p_*\iota\alpha(\partial\sigma) = p_*\iota d\alpha(\sigma), \tag{1.83}$$

so f_α is a differential character $f_\alpha \in \hat{H}^k(M, \mathbb{R}/\Lambda)$ with $\delta_1(f_\alpha) = d\alpha$. Write $i_2(\alpha) = f_\alpha$. For $\beta \in \Omega^k_{\mathrm{cl}}(M)_\Lambda$, by definition $\int_{z \in Z_k(M)} \beta \in \Lambda$, which maps to 0 under p. Therefore i_2 defines a homomorphism

$$\Omega^k(M)/\Omega^k_{\mathrm{cl}}(M)_\Lambda \xrightarrow{i_2} \hat{H}^k(M, \mathbb{R}/\Lambda), \quad \alpha \mapsto p_*\iota\alpha|_{Z_k(M)}. \tag{1.84}$$

If $i_2(\alpha) = 0$, namely if $\int_z \alpha \in \Lambda$ for every $z \in Z_k(M)$, then in particular for every $\partial\sigma \in B_k(M)$ we have $\int_{\partial\sigma} \alpha = \int_\sigma d\alpha = \iota d\alpha(\sigma) \in \Lambda$. By injectivity of $p_*\iota$, this implies $d\alpha = 0$. Thus α is closed and has periods in Λ. Hence the kernel of i_2 is $\Omega^k_{\mathrm{cl}}(M)_\Lambda$, and i_2 is injective. Moreover, for $f = i_2(\alpha)$ one can take $T_f = \iota\alpha$ as a representation, so $u_f = \delta_2(f) = 0$.

Proposition 1.16.

$$\delta_1 \circ i_2 = d. \tag{1.85}$$

Proposition 1.17. *There is an exact sequence*

$$0 \to \Omega^k(M)/\Omega^k_{\mathrm{cl}}(M)_\Lambda \xrightarrow{i_2} \hat{H}^k(M, \mathbb{R}/\Lambda) \xrightarrow{\delta_2} H^{k+1}(M, \Lambda) \to 0. \tag{1.86}$$

(Proof) The injectivity of i_2, the surjectivity of δ_2, and $\delta_2 \circ i_2 = 0$ have already been shown. Let $f \in \ker \delta_2$.

Thus for T_f satisfying $p_*T_f|_{Z_k(M)} = f$, suppose $\delta T_f = \iota\omega_f + r_*\delta e$. Then $\iota\omega_f = \delta(T_f - r_*e)$, and by de Rham's theorem $\omega_f = d\theta$. Hence $\delta(T_f - r_*e - \iota\theta) = 0$, so a cocycle $a = T_f - r_*e - \iota\theta \in Z^k(M,\mathbb{R})$ is defined. Again by de Rham's theorem, there exist $\phi \in \Omega^k_{\mathrm{cl}}(M)$ and $\eta \in \Omega^{k-1}(M)$ such that $a = \iota\phi + \iota d\eta$. Therefore

$$T_f = \iota(\phi + \theta + d\eta) + r_*e. \tag{1.87}$$

Consequently

$$\delta T_f = \iota d(\phi + \theta) + r_*\delta e. \tag{1.88}$$

Thus $f = i_2(\phi + \theta) = p_*\iota(\phi + \theta)|_{Z_k(M)}$. □

Introduce the following.

Definition 1.18.

$$R^k(M,\Lambda) := \{(\omega, u) \in \Omega^k_{\mathrm{cl}}(M)_\Lambda \times H^k(M,\Lambda) | r_*u = [\iota\omega]\}, \tag{1.89}$$

$$R^*(M,\Lambda) := \oplus R^k(M,\Lambda). \tag{1.90}$$

Proposition 1.19. *The group $R^*(M,\Lambda)$ has a ring structure with respect to the product*

$$(u,\omega)\cdot(v,\phi) = (u \cup v, \omega \wedge \phi). \tag{1.91}$$

(Proof)

$$r_*(u \cup v) = r_*u \cup r_*v = [\iota\omega] \cup [\iota\phi] = \iota[\omega] \cup \iota[\phi] = \iota([\omega] \cup [\phi]) = \iota[\omega \wedge \phi] = [\iota(\omega \wedge \phi)]. \quad \square \tag{1.92}$$

Proposition 1.20. *There is an exact sequence*

$$0 \to H^k(M,\mathbb{R})/r_*[H^k(M,\Lambda)] \xrightarrow{i_{12}} \hat{H}^k(M,\mathbb{R}/\Lambda) \xrightarrow{(\delta_1,\delta_2)} R^{k+1}(M,\Lambda) \to 0. \tag{1.93}$$

(Proof) The surjectivity of (δ_1, δ_2) follows from Corollary 1.13. For $[T] \in H^k(M,\mathbb{R})$, define i_{12} by

$$f = i_{12}([T]) := p_*T|_{Z_k(M)}. \tag{1.94}$$

Since $\delta S(z \in Z_k(M)) = 0$, this does not depend on the choice of T. Since $\delta T = 0$, $p_*T(\partial\sigma) = p_*\delta T(\sigma) = 0$, and hence $f \in \hat{H}^k(M,\mathbb{R}/\Lambda)$ with $\omega_f = 0$ and $u_f = 0$. Therefore $(\delta_1,\delta_2)\circ i_{12} = 0$. If $i_{12}([T]) = 0$, then $p_*T|_{Z_k(M)} = 0$. By (1.20), the projectivity of B_{k-1}, and the injectivity of $\mathbb{R}$, there exists $b \in \mathrm{Hom}(C_{k-1}(M),\mathbb{R})$ such that $p_*T = p_*b \circ \partial$. Thus there exists $c \in C^k(M,\Lambda)$ such that

$$T = \delta b + r_*c. \tag{1.95}$$

Since $r_*\delta c = \delta T = 0$, the cochain c is a cocycle and defines $[c] \in H^k(M,\Lambda)$. Since $[T] = r_*[c]$, the kernel of i_{12} is $r_*[H^k(M,\Lambda)]$.

Conversely, suppose $f \in \hat{H}^k(M,\mathbb{R}/\Lambda)$ satisfies $\delta_1(f) = 0$ and $\delta_2(f) = 0$. Then a representation $f = p_*T_f|_{Z_k(M)}$ of f satisfies $\delta T_f = r_*\delta e$ for some $e \in C^{k-1}(M,\Lambda)$. Therefore T_f determines a cohomology class $[T_f] \in H^k(M,\mathbb{R})$, and $i_{12}([T_f]) = f$. □

The following has been proved.

Theorem 1.21. *There are exact sequences*

$$0 \to H^k(M,\mathbb{R}/\Lambda) \xrightarrow{i_1} \hat{H}^k(M,\mathbb{R}/\Lambda) \xrightarrow{\delta_1} \Omega^{k+1}_{\mathrm{cl}}(M)_\Lambda \to 0, \tag{1.96}$$

$$0 \to \Omega^k(M)/\Omega^k_{\mathrm{cl}}(M)_\Lambda \xrightarrow{i_2} \hat{H}^k(M,\mathbb{R}/\Lambda) \xrightarrow{\delta_2} H^{k+1}(M,\Lambda) \to 0, \tag{1.97}$$

$$0 \to H^k(M,\mathbb{R})/r_*[H^k(M,\Lambda)] \xrightarrow{i_{12}} \hat{H}^k(M,\mathbb{R}/\Lambda) \xrightarrow{(\delta_1,\delta_2)} R^{k+1}(M,\Lambda) \to 0. \tag{1.98}$$

Moreover, the following hold. The maps i_1, i_2, i_{12} *are defined by*

$$i_1([s]) = s|_{Z_k(M)}, \tag{1.99}$$

$$i_2(\alpha) = p_*\iota\alpha|_{Z_k(M)}, \tag{1.100}$$

$$i_{12}([T]) = p_*T|_{Z_k(M)}. \tag{1.101}$$

If $T_f \in C^k(M,\mathbb{R})$ *is a representation of* $f \in \hat{H}^k(M,\mathbb{R}/\Lambda)$, $f = p_*T_f|_{Z_k(M)}$, *then*

$$\delta T_f = \iota\omega_f - r_*c_f, \quad \omega_f \in \Omega^{k+1}_{\mathrm{cl}}(M)_\Lambda, \quad c_f \in Z^{k+1}(M,\Lambda). \tag{1.102}$$

The maps δ_1, δ_2 *are defined by*

$$\delta_1(f) = \omega_f, \tag{1.103}$$

$$\delta_2(f) = [c_f]. \tag{1.104}$$

The following identities hold:

$$\delta_1 \circ i_2 = d, \tag{1.105}$$

$$\delta_2 \circ i_1 = -B. \tag{1.106}$$

Here B *is the Bockstein homomorphism.*

1.5.1 Example: Euler class of a circle bundle

Let $SO(2) \to E \xrightarrow{\pi} M$ be a circle bundle with connection θ, and let Ω be its curvature form. Since $\frac{1}{2\pi}\Omega$ represents the Euler class, $\frac{1}{2\pi}\Omega \in \Omega^2_{\mathrm{cl}}(M)_\mathbb{Z}$. For a smooth closed path $\gamma : S^1 \to M$, the $SO(2)$ holonomy $H(\gamma)$ is defined. Define a differential character on smooth closed paths by $e^{2\pi i}\hat{\chi}(\gamma) = H(\gamma)$. To define a differential character for an arbitrary cycle $x \in Z_1(M)$, note that there exists a smooth closed path $\gamma : S^1 \to M$ such that $x = \gamma + \partial y$ for some y[2]. Then define

$$\hat{\chi}(x) := \hat{\chi}(\gamma) + \frac{1}{2\pi} p\left(\int_y \Omega\right) = \hat{\chi}(\gamma) + \frac{1}{2\pi} p_*\iota\Omega(y). \tag{1.109}$$

In this example, $\delta_1(\hat{\chi}) = \frac{1}{2\pi}\Omega$ and $\delta_2(\hat{\chi}) = \chi$ (the Euler class).

[2] An element of $x \in Z_1(M)$ is, by definition, a 1-chain without boundary, and a k-chain is an integer linear combination of nondegenerate cubes

$$\sigma : I^k \to M, \quad \text{smooth}, \tag{1.107}$$

namely

$$x = \sum_i n_i\sigma_i, \quad n_i \in \mathbb{Z}, \quad \partial x = \sum_i n_i \partial\sigma_i = 0. \tag{1.108}$$

Therefore x **itself is not necessarily smooth**. On the other hand, it agrees with some smooth closed path $\gamma : S^1 \to M$ up to a boundary, so it can be written as $x = \gamma + \partial y$.

1.6 Graded ring structure

As already checked in [3], suppose the subdivision operator $\Delta : C_k(M) \to C_k(M)$ is shown, step by step, to have the following properties:

- In the limit of subdivision, the cup product and wedge product agree:

$$\lim_{n\to\infty} (\Delta^*)^n(\iota\omega \cup \iota\theta) = \iota(\omega \wedge \theta). \tag{1.110}$$

- For $\omega_1 \in \Omega^{l_1}(M)$ and $\omega_2 \in \Omega^{l_2}(M)$, the following infinite sum converges:

$$E : \Omega^{l_1}(M) \times \Omega^{l_2}(M) \to C^{l_1+l_2-1}(M, \mathbb{R}), \quad \text{bilinear}, \tag{1.111}$$

$$E(\omega \otimes \theta)(x) := -\sum_{i=0}^{\infty} (\iota\omega \cup \iota\theta)(\psi\Delta^i x). \tag{1.112}$$

  Here ψ is a chain homotopy between Δ and 1:

$$1 - \Delta = \partial\psi + \psi\partial. \tag{1.113}$$

Then, with

$$E(\omega \otimes \theta) := -\sum_{i=0}^{\infty} (\Delta^*)^i \psi^*(\iota\omega \cup \iota\theta), \tag{1.114}$$

the map E is a chain homotopy connecting $\cup$ and $\wedge$:

$$\iota(\omega_1 \wedge \omega_2) - \iota\omega_1 \cup \iota\omega_2 = \delta E(\omega_1 \otimes \omega_2) + Ed(\omega_1 \otimes \omega_2) \tag{1.115}$$

$$= \delta E(\omega_1, \omega_2) + E(d\omega_1, \omega_2) + (-1)^{l_1} E(\omega_1, d\omega_2). \tag{1.116}$$

In particular, for closed forms ω_1, ω_2,

$$\iota(\omega_1 \wedge \omega_2) - \iota\omega_1 \cup \iota\omega_2 = \delta E(\omega_1 \otimes \omega_2), \quad \omega_1 \in \Omega^{l_1}_{\mathrm{cl}}(M), \omega_2 \in \Omega^{l_2}_{\mathrm{cl}}(M). \tag{1.117}$$

Moreover,

$$(\Delta^*)^n E(\omega_1 \otimes \omega_2) = E(\omega_1 \otimes \omega_2) + \sum_{i=0}^{n-1} (\Delta^*)^i \psi^*(\iota\omega_1 \cup \iota\omega_2), \tag{1.118}$$

so

$$\lim_{n\to\infty} (\Delta^*)^n E(\omega_1 \otimes \omega_2) = 0. \tag{1.119}$$

We make several preparations.

- Since $r : \Lambda \to \mathbb{R}$ is compatible with products, $r(xy) = r(x)r(y)$, the map r_* is compatible with the cup product on the cochain level:

$$r_* c_1 \cup r_* c_2 = r_*(c_1 \cup c_2). \tag{1.120}$$

- On the cochain level, the cup product is neither graded commutative nor associative. However, these identities hold in cohomology. Write the chain homotopies as

$$H : C^*(M, A) \otimes C^*(M, A) \to C^{*-1}(M, A), \tag{1.121}$$

$$f \cup g - (-1)^{\deg(f)\deg(g)} g \cup f \tag{1.122}$$

$$= \delta H(f \otimes g) + H(\delta(f \otimes g)) \tag{1.123}$$

$$= \delta H(f, g) + H(\delta f, g) + (-1)^{\deg(f)} H(f, \delta g), \tag{1.124}$$

$$K : C^*(M,A) \otimes C^*(M,A) \otimes C^*(M,A) \to C^{*-1}(M,A), \tag{1.125}$$
$$(f \cup g) \cup h - f \cup (g \cup h) \tag{1.126}$$
$$= \delta K(f \otimes g \otimes h) + K(\delta(f \otimes g \otimes h)) \tag{1.127}$$
$$= \delta K(f,g,h) + K(\delta f, g, h) + (-1)^{\deg(f)} K(f, \delta g, h) + (-1)^{\deg(f)+\deg(g)} K(f,g,\delta h), \tag{1.128}$$

and so on.

- On the other hand, the noncommutativity of the cup product of differential forms can be written using E. From (1.116),

$$\iota\omega_1 \cup \iota\omega_2 - (-1)^{l_1 l_2} \iota\omega_2 \cup \iota\omega_1 \tag{1.129}$$
$$= -\delta E(\omega_1 \otimes \omega_2 - (-1)^{l_1 l_2} \omega_2 \otimes \omega_1) - Ed(\omega_1 \otimes \omega_2 - (-1)^{l_1 l_2} \omega_2 \otimes \omega_1) \tag{1.130}$$
$$= -(\delta E + E\delta)(\omega_1 \otimes \omega_2 - (-1)^{l_1 l_2} \omega_2 \otimes \omega_1). \tag{1.131}$$

Thus

$$H(\iota\omega_1 \otimes \iota\omega_2) = -E(\omega_1 \otimes \omega_2 - (-1)^{l_1 l_2} \omega_2 \otimes \omega_1). \tag{1.132}$$

In other words,

$$H \circ (\iota \otimes \iota) = -E \circ (\mathrm{id} - (-1)^{l_1 l_2} \tau), \tag{1.133}$$

where

$$\tau : \Omega^* \otimes \Omega^* \to \Omega^* \otimes \Omega^*, \quad \omega_1 \otimes \omega_2 \mapsto \omega_2 \otimes \omega_1 \tag{1.134}$$

is the braiding.

- The failure of associativity for differential forms can also be expressed using E. Let $\omega_1, \omega_2, \omega_3$ be forms of degrees l_1, l_2, l_3, respectively. Then

$$(\iota\omega_1 \cup \iota\omega_2) \cup \iota\omega_3 - \iota\omega_1 \cup (\iota\omega_2 \cup \iota\omega_3) \tag{1.135}$$
$$= (\iota(\omega_1 \wedge \omega_2) - (\delta E + Ed)(\omega_1 \otimes \omega_2)) \cup \iota\omega_3 - \iota\omega_1 \cup (\iota(\omega_2 \wedge \omega_3) - (\delta E + Ed)(\omega_2 \otimes \omega_3)) \tag{1.136}$$
$$= \iota(\omega_1 \wedge \omega_2 \wedge \omega_3) - (\delta E + Ed)((\omega_1 \wedge \omega_2) \otimes \omega_3) \tag{1.137}$$
$$- (\iota(\omega_1 \wedge \omega_2 \wedge \omega_3) - (\delta E + Ed)(\omega_1 \otimes (\omega_2 \wedge \omega_3))) \tag{1.138}$$
$$- ((\delta E + Ed)(\omega_1 \otimes \omega_2)) \cup \iota\omega_3 + \iota\omega_1 \cup (\delta E + Ed)(\omega_2 \otimes \omega_3) \tag{1.139}$$
$$= -(\delta E + Ed)((\omega_1 \wedge \omega_2) \otimes \omega_3 - \omega_1 \otimes (\omega_2 \wedge \omega_3)) \tag{1.140}$$
$$- ((\delta E + Ed)(\omega_1 \otimes \omega_2)) \cup \iota\omega_3 + \iota\omega_1 \cup (\delta E + Ed)(\omega_2 \otimes \omega_3). \tag{1.141}$$

The first line can be written as

$$-(\delta E + Ed)((\omega_1 \wedge \omega_2) \otimes \omega_3 - \omega_1 \otimes (\omega_2 \wedge \omega_3)) = (\delta A + Ad)(\omega_1 \otimes \omega_2 \otimes \omega_3), \tag{1.142}$$

$$A = -E \circ (\wedge \otimes \mathrm{id} - \mathrm{id} \otimes \wedge). \tag{1.143}$$

The first term of the second line is

$$((\delta E + Ed)(\omega_1 \otimes \omega_2)) \cup \iota\omega_3 \tag{1.144}$$
$$= (\delta E(\omega_1 \otimes \omega_2)) \cup \iota\omega_3 + E(d(\omega_1 \otimes \omega_2)) \cup \iota\omega_3 \tag{1.145}$$
$$= \delta(E(\omega_1 \otimes \omega_2) \cup \iota\omega_3) - (-1)^{l_1+l_2-1} E(\omega_1 \otimes \omega_2) \cup \iota d\omega_3 + E(d(\omega_1 \otimes \omega_2)) \cup \iota\omega_3 \tag{1.146}$$
$$= \delta(E(\omega_1 \otimes \omega_2) \cup \iota\omega_3) + (-1)^{l_1+l_2} E(\omega_1 \otimes \omega_2) \cup \iota d\omega_3 + E(d(\omega_1 \otimes \omega_2)) \cup \iota\omega_3 \tag{1.147}$$
$$= (\delta B + Bd)(\omega_1 \otimes \omega_2 \otimes \omega_3), \tag{1.148}$$

where

$$B = \cup \circ (E \otimes \iota). \tag{1.149}$$

Similarly, the second term of the second line is

$$\begin{aligned}
&\iota\omega_1 \cup (\delta E + Ed)(\omega_2 \otimes \omega_3) && (1.150)\\
&= (-1)^{l_1}\delta(\iota\omega_1 \cup E(\omega_2 \otimes \omega_3)) - (-1)^{l_1}\iota d\omega_1 \cup E(\omega_2 \otimes \omega_3) + \iota\omega_1 \cup Ed(\omega_2 \otimes \omega_3) && (1.151)\\
&= (-1)^{\deg(\omega_1)}\delta(\iota\omega_1 \cup E(\omega_2 \otimes \omega_3)) + (-1)^{\deg(d\omega_1)}\iota d\omega_1 \cup E(\omega_2 \otimes \omega_3) + \iota\omega_1 \cup Ed(\omega_2 \otimes \omega_3) && (1.152)\\
&= (\delta C + Cd)(\omega_1 \otimes \omega_2 \otimes \omega_3). && (1.153)
\end{aligned}$$

Here we set

$$C(\omega_1 \otimes \omega_2 \otimes \omega_3) = (-1)^{\deg(\omega_1)}(\cup \circ \iota \otimes E)(\omega_1 \otimes \omega_2 \otimes \omega_3), \tag{1.154}$$

that is,

$$C = \cup \circ (((-1)^{\deg} \circ \iota) \otimes E). \tag{1.155}$$

Indeed,

$$\begin{aligned}
&(\delta C + Cd)(\omega_1 \otimes \omega_2 \otimes \omega_3) && (1.156)\\
&= \delta C(\omega_1 \otimes \omega_2 \otimes \omega_3) + C(d\omega_1 \otimes \omega_2 \otimes \omega_3) + (-1)^{\deg(\omega_1)}C(\omega_1 \otimes d(\omega_2 \otimes \omega_3)) && (1.157)\\
&= (-1)^{\deg(\omega_1)}\delta(\cup \circ \iota \otimes E)(\omega_1 \otimes \omega_2 \otimes \omega_3) && (1.158)\\
&\quad + (-1)^{\deg(d\omega_1)}(\cup \circ \iota \otimes E)(d\omega_1 \otimes \omega_2 \otimes \omega_3) + (\cup \circ \iota \otimes E)(\omega_1 \otimes d(\omega_2 \otimes \omega_3)) && (1.159)
\end{aligned}$$

as required. Therefore

$$(\iota\omega_1 \cup \iota\omega_2) \cup \iota\omega_3 - \iota\omega_1 \cup (\iota\omega_2 \cup \iota\omega_3) = (\delta D + Dd)(\omega_1 \otimes \omega_2 \otimes \omega_3), \tag{1.160}$$

$$D = A - B + C = -E \circ (\wedge \otimes \mathrm{id} - \mathrm{id} \otimes \wedge) - \cup \circ (E \otimes \iota) + \cup \circ (((-1)^{\deg} \circ \iota) \otimes E). \tag{1.161}$$

In other words,

$$K \circ (\iota \otimes \iota \otimes \iota) = -E \circ (\wedge \otimes \mathrm{id} - \mathrm{id} \otimes \wedge) - \cup \circ (E \otimes \iota) + \cup \circ (((-1)^{\deg} \circ \iota) \otimes E). \tag{1.162}$$

In variables, this gives

$$\begin{aligned}
K(\iota\omega_1 \otimes \iota\omega_2 \otimes \iota\omega_3) &= -E((\omega_1 \wedge \omega_2) \otimes \omega_3) + E(\omega_1 \otimes (\omega_2 \wedge \omega_3)) && (1.163)\\
&\quad - E(\omega_1 \otimes \omega_2) \cup \iota\omega_3 + (-1)^{\deg(\omega_1)}\iota\omega_1 \cup E(\omega_2 \otimes \omega_3). && (1.164)
\end{aligned}$$

Definition 1.22. For $f \in \hat{H}^k(M, \mathbb{R}/\Lambda)$ and $g \in \hat{H}^l(M, \mathbb{R}/\Lambda)$, choose representations $f = p_* T_f|_{Z_k(M)}$ and $g = p_* T_g|_{Z_l(M)}$. Define

$$f \star g := p_* \left(T_f \cup \iota\omega_g - (-1)^k \iota\omega_f \cup T_g - T_f \cup \delta T_g + E(\omega_f, \omega_g)\right)|_{Z_{k+l+1}}. \tag{1.165}$$

Thus $f \star g$ is the differential character represented by

$$T_{f\star g} = T_f \cup \iota\omega_g - (-1)^k \iota\omega_f \cup T_g - T_f \cup \delta T_g + E(\omega_f, \omega_g). \tag{1.166}$$

Choosing a representative of $u_g = [c_g]$, one can also write

$$T_{f\star g} = T_f \cup r_* c_g - (-1)^k \iota\omega_f \cup T_g + E(\omega_f, \omega_g). \tag{1.167}$$

We check the properties in order. Below, equalities up to $\operatorname{im}\delta$ and $\operatorname{im} r_*$ are denoted by $\sim_\delta$ and $\sim_r$, respectively, and equality up to $\operatorname{im}\delta + \operatorname{im} r_*$ is denoted by $\sim_{\delta,r}$. (Below there is some fluctuation between the notations $E(\omega_1, \omega_2)$ and $E(\omega_1 \otimes \omega_2)$, but they have the same meaning.)

Proposition 1.23. *The product $f \star g$ is independent of the choices of the representations T_f, T_g.*

(Proof) The ambiguities of T_f, T_g are

$$T_f \mapsto T_f + r_* d_f + \delta s_f, \quad d_f \in C^k(M, \Lambda), \quad s_f \in C^{k-1}(M, \mathbb{R}), \tag{1.168}$$
$$T_g \mapsto T_g + r_* d_g + \delta s_g, \quad d_g \in C^l(M, \Lambda), \quad s_g \in C^{l-1}(M, \mathbb{R}). \tag{1.169}$$

Under this change,

$$\begin{aligned}
T_{f\star g} \mapsto & (T_f + r_* d_f + \delta s_f) \cup \iota\omega_g - (-1)^k \iota\omega_f \cup (T_g + r_* d_g + \delta s_g) && (1.170)\\
& - (T_f + r_* d_f + \delta s_f) \cup \delta(T_g + r_* d_g + \delta s_g) + E(\omega_f, \omega_g) && (1.171)\\
= & T_{f\star g} + (r_* d_f + \delta s_f) \cup \iota\omega_g - (-1)^k \iota\omega_f \cup (r_* d_g + \delta s_g) && (1.172)\\
& - T_f \cup \delta(r_* d_g + \delta s_g) - (r_* d_f + \delta s_f) \cup \delta(T_g + r_* d_g + \delta s_g) && (1.173)\\
= & T_{f\star g} + r_* d_f \cup \iota\omega_g + \delta(s_f \cup \iota\omega_g) - (-1)^k \iota\omega_f \cup r_* d_g + \delta(\iota\omega_f \cup s_g) && (1.174)\\
& - T_f \cup \delta r_* d_g - (r_* d_f + \delta s_f) \cup (\delta T_g + \delta r_* d_g) && (1.175)\\
= & T_{f\star g} + r_* d_f \cup (\iota\omega_g - \delta T_g) + \delta(s_f \cup \iota\omega_g) - (-1)^k \iota\omega_f \cup r_* d_g + \delta(\iota\omega_f \cup s_g) && (1.176)\\
& - (-1)^k (\delta(T_f \cup r_* d_g) - (\iota\omega_f - r_* c_f) \cup r_* d_g) - (r_* d_f + \delta s_f) \cup \delta r_* d_g - \delta s_f \cup \delta T_g && (1.177)\\
= & T_{f\star g} + r_* d_f \cup r_* c_g + \delta(s_f \cup \iota\omega_g) + \delta(\iota\omega_f \cup s_g) && (1.178)\\
& - (-1)^k \delta(T_f \cup r_* d_g) - (-1)^k r_* c_f \cup r_* d_g - (r_* d_f + \delta s_f) \cup \delta r_* d_g - \delta s_f \cup \delta T_g && (1.179)\\
= & T_{f\star g} + r_* \left(d_f \cup c_g - (-1)^k c_f \cup d_g - d_f \cup \delta d_g\right) && (1.180)\\
& + \delta \left(s_f \cup \iota\omega_g + \iota\omega_f \cup s_g - (-1)^k T_f \cup r_* d_g - s_f \cup \delta r_* d_g - s_f \cup \delta T_g\right). && (1.181)
\end{aligned}$$

Here we used

$$\begin{aligned}
\delta(T_f \cup r_* d_g) &= \delta T_f \cup r_* d_g + (-1)^k T_f \cup \delta r_* d_g && (1.182)\\
&= (\iota\omega_f - r_* c_f) \cup r_* d_g + (-1)^k T_f \cup \delta r_* d_g. && (1.183)
\end{aligned}$$

Thus the change of $T_{f\star g}$ is an ambiguity of the representation, and $f \star g$ is well-defined.

Also note that, in the expression (1.167), the ambiguity $c_g \mapsto c_g + \delta d_g$ of the representative of $u_g = [c_g]$ is part of the ambiguity of T_g. □

Proposition 1.24. *The product $f \star g$ is indeed a differential character:*

$$f \star g \in \hat{H}^{k+l+1}(M, \mathbb{R}/\Lambda). \tag{1.184}$$

Namely, there exists $\omega \in \Omega^{k+l+1}(M)$ such that $(f \star g) \circ \partial = p_* \iota\omega$.

(Proof) It is enough to check this at the level of representations.

$$\begin{aligned}
& \delta \left(T_f \cup r_* c_g - (-1)^k \iota\omega_f \cup T_g + E(\omega_f, \omega_g)\right) && (1.185)\\
& = \delta T_f \cup r_* c_g + \iota\omega_f \cup \delta T_g + \delta E(\omega_f, \omega_g) && (1.186)\\
& = (\iota\omega_f - r_* c_f) \cup r_* c_g + \iota\omega_f \cup (\iota\omega_g - r_* c_g) + \iota(\omega_f \wedge \omega_g) - \iota\omega_f \cup \iota\omega_g && (1.187)\\
& = \iota(\omega_f \wedge \omega_g) - r_*(c_f \cup c_g). && (1.188)
\end{aligned}$$

Notice that

$$r_* c_f \cup r_* c_g = r_*(c_f \cup c_g). \tag{1.189}$$

Thus $f \star g$ is a differential character with

$$\delta_1(f \star g) = \omega_{f\star g} = \omega_f \wedge \omega_g, \tag{1.190}$$
$$\delta_2(f \star g) = u_{f\star g} = u_f \cup u_g. \tag{1.191}$$

□

Proposition 1.25.

$$f \star g = (-1)^{(k+1)(l+1)} g \star f. \tag{1.192}$$

(Proof) Compare the expression

$$T_{g\star f} = T_g \cup r_* c_f - (-1)^l \iota\omega_g \cup T_f + E(\omega_g, \omega_f) \tag{1.193}$$

with

$$\begin{aligned} T_f \cup r_* c_g &\sim_\delta (-1)^{k(l+1)} r_* c_g \cup T_f + H(\delta T_f \otimes r_* c_g) && (1.194)\\ &\sim_r (-1)^{k(l+1)} r_* c_g \cup T_f + H(\iota\omega_f \otimes r_* c_g), && (1.195)\end{aligned}$$

and

$$\begin{aligned} \iota\omega_f \cup T_g &\sim_\delta (-1)^{(k+1)l} T_g \cup \iota\omega_f + (-1)^{k+1} H(\iota\omega_f \otimes \delta T_g) && (1.196)\\ &= (-1)^{(k+1)l} T_g \cup \iota\omega_f + (-1)^{k+1} H(\iota\omega_f \otimes (\iota\omega_g - r_* c_g)). && (1.197)\end{aligned}$$

Then

$$\begin{aligned} &T_{f\star g} - (-1)^{(k+1)(l+1)} T_{g\star f} && (1.198)\\ &\sim_{\delta,r} (-1)^{k(l+1)} r_* c_g \cup T_f + H(\iota\omega_f \otimes r_* c_g) && (1.199)\\ &\quad - (-1)^k ((-1)^{(k+1)l} T_g \cup \iota\omega_f + (-1)^{k+1} H(\iota\omega_f \otimes (\iota\omega_g - r_* c_g))) + E(\omega_f, \omega_g) && (1.200)\\ &\quad - (-1)^{(k+1)(l+1)} (T_g \cup r_* c_f - (-1)^l \iota\omega_g \cup T_f + E(\omega_g, \omega_f)) && (1.201)\\ &= -(-1)^{k(l+1)} \delta T_g \cup T_f + (-1)^{(k+1)(l+1)} T_g \cup \delta T_f + H(\iota\omega_f \otimes r_* c_g) && (1.202)\\ &\quad + H(\iota\omega_f \otimes (\iota\omega_g - r_* c_g)) + E(\omega_f, \omega_g) - (-1)^{(k+1)(l+1)} E(\omega_g, \omega_f) && (1.203)\\ &\sim_\delta H(\iota\omega_f \otimes \iota\omega_g) + E(\omega_f, \omega_g) - (-1)^{(k+1)(l+1)} E(\omega_g, \omega_f). && (1.204)\end{aligned}$$

This vanishes by (1.132). □

Proposition 1.26.

$$(f \star g) \star h = f \star (g \star h). \tag{1.205}$$

(Proof) Let f, g, h be differential characters of degrees k, l, m, with representations T_f, T_g, T_h.

$$\begin{aligned} T_{(f\star g)\star h} &= T_{f\star g} \cup r_* c_h - (-1)^{k+l+1} \iota\omega_{f\star g} \cup T_h + E(\omega_{f\star g}, \omega_h) && (1.206)\\ &= \big(T_f \cup r_* c_g - (-1)^k \iota\omega_f \cup T_g + E(\omega_f, \omega_g)\big) \cup r_* c_h - (-1)^{k+l+1} \iota(\omega_f \wedge \omega_g) \cup T_h + E(\omega_f \wedge \omega_g, \omega_h) && (1.207)\\ &= (T_f \cup r_* c_g) \cup r_* c_h - (-1)^k (\iota\omega_f \cup T_g) \cup r_* c_h + E(\omega_f, \omega_g) \cup r_* c_h && (1.208)\\ &\quad - (-1)^{k+l+1} \iota(\omega_f \wedge \omega_g) \cup T_h + E(\omega_f \wedge \omega_g, \omega_h). && (1.209)\end{aligned}$$

On the other hand,

$$\begin{aligned} T_{f\star(g\star h)} &= T_f \cup r_* c_{g\star h} - (-1)^k \iota\omega_f \cup T_{g\star h} + E(\omega_f, \omega_{g\star h}) && (1.210)\\ &= T_f \cup r_* (c_g \cup c_h) - (-1)^k \iota\omega_f \cup \big(T_g \cup r_* c_h - (-1)^l \iota\omega_g \cup T_h + E(\omega_g, \omega_h)\big) + E(\omega_f, \omega_g \wedge \omega_h) && (1.211)\\ &= T_f \cup (r_* c_g \cup r_* c_h) - (-1)^k \iota\omega_f \cup (T_g \cup r_* c_h) + (-1)^{k+l} \iota\omega_f \cup (\iota\omega_g \cup T_h) && (1.212)\\ &\quad - (-1)^k \iota\omega_f \cup E(\omega_g, \omega_h) + E(\omega_f, \omega_g \wedge \omega_h). && (1.213)\end{aligned}$$

Compare these. Using

$$\iota\omega_f \cup (\iota\omega_g \cup T_h) = (\iota\omega_f \cup \iota\omega_g) \cup T_h - \delta K(\iota\omega_f \otimes \iota\omega_g \otimes T_h) - (-1)^{k+l+2} K(\iota\omega_f \otimes \iota\omega_g \otimes \delta T_h), \tag{1.214}$$

we get

$$\begin{aligned}
&T_{(f\star g)\star h} - T_{f\star(g\star h)} && (1.215)\\
&= E(\omega_f, \omega_g) \cup r_* c_h + (-1)^{k+l} \iota(\omega_f \wedge \omega_g) \cup T_h + E(\omega_f \wedge \omega_g, \omega_h) && (1.216)\\
&\quad - (-1)^{k+l}((\iota\omega_f \cup \iota\omega_g) \cup T_h - \delta K(\iota\omega_f \otimes \iota\omega_g \otimes T_h) - (-1)^{k+l+2} K(\iota\omega_f \otimes \iota\omega_g \otimes \delta T_h)) && (1.217)\\
&\quad + (-1)^k \iota\omega_f \cup E(\omega_g, \omega_h) - E(\omega_f, \omega_g \wedge \omega_h) && (1.218)\\
&\quad + \delta K(T_f \otimes r_* c_g \otimes r_* c_h) + K(\delta T_f \otimes r_* c_g \otimes r_* c_h) && (1.219)\\
&\quad - (-1)^k (\delta K(\iota\omega_f \otimes T_g \otimes r_* c_h) + (-1)^{k+1} K(\iota\omega_f \otimes \delta T_g \otimes r_* c_h)). && (1.220)
\end{aligned}$$

Further transforming up to $\operatorname{im}\delta, \operatorname{im} r_*$,

$$\begin{aligned}
&T_{(f\star g)\star h} - T_{f\star(g\star h)} && (1.221)\\
&\sim_\delta E(\omega_f, \omega_g) \cup r_* c_h + (-1)^{k+l} \iota(\omega_f \wedge \omega_g) \cup T_h + E(\omega_f \wedge \omega_g, \omega_h) && (1.222)\\
&\quad - (-1)^{k+l} (\iota\omega_f \cup \iota\omega_g) \cup T_h + K(\iota\omega_f \otimes \iota\omega_g \otimes \delta T_h) && (1.223)\\
&\quad + (-1)^k \iota\omega_f \cup E(\omega_g, \omega_h) - E(\omega_f, \omega_g \wedge \omega_h) && (1.224)\\
&\quad + K(\delta T_f \otimes r_* c_g \otimes r_* c_h) + K(\iota\omega_f \otimes \delta T_g \otimes r_* c_h) && (1.225)\\
&\sim_\delta E(\omega_f, \omega_g) \cup r_* c_h + (-1)^{k+l} \delta E(\omega_f \otimes \omega_g) \cup T_h + E(\omega_f \wedge \omega_g, \omega_h) && (1.226)\\
&\quad + K(\iota\omega_f \otimes \iota\omega_g \otimes (\iota\omega_h - r_* c_h)) && (1.227)\\
&\quad + (-1)^k \iota\omega_f \cup E(\omega_g, \omega_h) - E(\omega_f, \omega_g \wedge \omega_h) && (1.228)\\
&\quad + K((\iota\omega_f - r_* c_f) \otimes r_* c_g \otimes r_* c_h) + K(\iota\omega_f \otimes (\iota\omega_g - r_* c_g) \otimes r_* c_h) && (1.229)\\
&\sim_\delta E(\omega_f, \omega_g) \cup r_* c_h + (-1)^{k+l}(-(-1)^{k+l+2} E(\omega_f \otimes \omega_g) \cup \delta T_h) + E(\omega_f \wedge \omega_g, \omega_h) && (1.230)\\
&\quad + (-1)^k \iota\omega_f \cup E(\omega_g, \omega_h) - E(\omega_f, \omega_g \wedge \omega_h) && (1.231)\\
&\quad + K(\iota\omega_f \otimes \iota\omega_g \otimes \iota\omega_h) - K(r_* c_f \otimes r_* c_g \otimes r_* c_h) && (1.232)\\
&\sim_\delta E(\omega_f, \omega_g) \cup \iota\omega_h + E(\omega_f \wedge \omega_g, \omega_h) + (-1)^k \iota\omega_f \cup E(\omega_g, \omega_h) - E(\omega_f, \omega_g \wedge \omega_h) && (1.233)\\
&\quad + K(\iota\omega_f \otimes \iota\omega_g \otimes \iota\omega_h) - K(r_* c_f \otimes r_* c_g \otimes r_* c_h). && (1.234)
\end{aligned}$$

Here, because $r : \Lambda \to \mathbb{R}$ is compatible with products,

$$K(r_* c_f \otimes r_* c_g \otimes r_* c_h) = r_* K(c_f \otimes c_g \otimes c_h) \tag{1.235}$$

lies in $\operatorname{im} r_*$. The remaining terms cancel by (1.164). □

The above also shows:

Proposition 1.27.

$$(\delta_1, \delta_2) : \hat{H}^*(M, \mathbb{R}/\Lambda) \to R^*(M, \Lambda) \tag{1.236}$$

is a ring homomorphism.

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