Computation of the Chern-Number Density in a Dirac Model

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The Chern number is defined by

$$Ch_n = \int ch_n, \tag{1}$$

$$ch_n = \frac{1}{n!}\left(\frac{i}{2\pi}\right)^n \mathrm{tr}[F^n]. \tag{2}$$

If the occupied states of a Hamiltonian H are written as $\Phi = (u_1, \dots, u_n)$, the Berry connection is

$$A = \Phi^\dagger d\Phi, \tag{3}$$

$$F = dA + A^2 = d\Phi^\dagger(1-\Phi\Phi^\dagger)d\Phi. \tag{4}$$

Using

$$(1-P)d\Phi = dP\,\Phi, \quad d\Phi^\dagger(1-P) = \Phi^\dagger dP, \tag{5}$$

we have

$$F = d\Phi^\dagger(1-P)(1-P)d\Phi = \Phi^\dagger dP\,dP\,\Phi. \tag{6}$$

Thus the density can be written in terms of the projection as

$$ch_n = \frac{1}{n!}\left(\frac{i}{2\pi}\right)^n \mathrm{tr}[(PdPdP)^n] = \frac{1}{n!}\left(\frac{i}{2\pi}\right)^n \mathrm{tr}[P(dP)^{2n}], \tag{7}$$

where we used $PdPdP = dP(1-P)dP = dPdPP$. Using the flattened Hamiltonian

$$Q = \mathrm{sgn} H = 1 - 2P, \tag{8}$$

we obtain

$$ch_n = -\frac{1}{2^{2n+1}n!}\left(\frac{i}{2\pi}\right)^n \mathrm{tr}[Q(dQ)^{2n}]. \tag{9}$$

Here we used $\mathrm{tr}[(dQ)^{2n}] = d\mathrm{tr}[Q(dQ)^{2n-1}]$ and $QdQ = -dQQ$, which imply $\mathrm{tr}[Q(dQ)^{2n-1}] = 0$.

A Dirac model is a model whose Hamiltonian can be written using mutually anticommuting gamma matrices

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}, \quad i, j = 1, \dots, 2n+1 \tag{10}$$

as

$$H = \sum_{i=1}^{2n+1} h_i \gamma_i. \tag{11}$$

Then the flattened Hamiltonian is

$$Q = \sum_{i=1}^{2n+1} \hat{h}_i \gamma_i, \quad \hat{h}_i = \frac{h_i}{|h|}, \quad |h| = \sqrt{\sum_i h_i^2}. \tag{12}$$

The Chern density is

$$ch_n = -\frac{1}{2^{2n+1}n!}\left(\frac{i}{2\pi}\right)^n \mathrm{tr}[Q(dQ)^{2n}] \tag{13}$$

$$= -\frac{1}{2^{2n+1}n!}\left(\frac{i}{2\pi}\right)^n \hat{h}_{i_0} d\hat{h}_{i_1}\cdots d\hat{h}_{i_{2n}} \mathrm{tr}[\gamma_{i_0}\gamma_{i_1}\cdots\gamma_{i_{2n}}]. \tag{14}$$

For a choice of $2^n \times 2^n$ gamma matrices, one has

$$\mathrm{tr}[\gamma_{i_0}\cdots\gamma_{i_{2n}}] \propto \pm 2^n i^n \epsilon_{i_0 i_1\cdots i_{2n}}, \tag{15}$$

where the sign depends on the definition of the gamma matrices. For example, define the $2^{n+1}\times 2^{n+1}$ gamma matrices recursively from $2^n \times 2^n$ gamma matrices by

$$\gamma_1^{(2)} = \sigma_1, \quad \gamma_2^{(2)} = \sigma_2, \quad \gamma_3^{(2)} = \sigma_3, \tag{16}$$

and

$$\gamma_i^{(2n+2)} = \begin{cases} \gamma_i^{(2n)} \otimes \sigma_1, & i = 1,\ldots,2n+1, \\ 1\otimes\sigma_2, & i = 2n+2, \\ 1\otimes\sigma_3, & i = 2n+3. \end{cases} \tag{17}$$

Then

$$\mathrm{tr}[\gamma_{i_0}^{(2n)}\cdots\gamma_{i_{2n}}^{(2n)}] = 2^n i^n \epsilon_{i_0 i_1\cdots i_{2n}}. \tag{18}$$

Adopting this convention, the Chern density becomes

$$ch_n = -\frac{1}{2^{n+1}n!}\left(\frac{-1}{2\pi}\right)^n \epsilon_{i_0 i_1\cdots i_{2n}} \hat{h}_{i_0} d\hat{h}_{i_1}\cdots d\hat{h}_{i_{2n}} \tag{19}$$

$$= (-1)^{n+1}\frac{(2n)!}{2^{2n+1}\pi^n n!}\,\omega_{2n}, \tag{20}$$

where

$$\omega_{2n} = \frac{1}{(2n)!}\epsilon_{i_0 i_1\cdots i_{2n}} \hat{h}_{i_0} d\hat{h}_{i_1}\cdots d\hat{h}_{i_{2n}} \tag{21}$$

is the volume form of the unit sphere S^{2n}. Since

$$A_{2n} = \frac{2^{2n+1}\pi^n n!}{(2n)!} \tag{22}$$

is the volume of the unit sphere S^{2n}, when $\hat{h}_i$ covers S^{2n} the Chern number is $(-1)^{n+1}$ times the covering number.

Moreover, only the dh_i part contributes to $d\hat{h}_i$. Indeed, when the gamma matrices are represented by $2^n \times 2^n$ matrices, the only terms that survive the trace are products containing all $2n+1$ gamma matrices. Therefore terms involving $d|h|$ do not contribute, and finally

$$ch_n = -\frac{1}{2^{n+1}n!}\left(\frac{-1}{2\pi}\right)^n \epsilon_{i_0 i_1\cdots i_{2n}} \frac{h_{i_0} dh_{i_1}\cdots dh_{i_{2n}}}{|h|^{2n+1}}. \tag{23}$$

Thus, for a given Dirac-model Hamiltonian, the difference of integrals of Chern–Simons forms can also be computed as

$$CS_{2n-1}[h(\lambda_{2n})] - CS_{2n-1}[h(\lambda_0)] = \int_{T^{2n-1}\times[\lambda_0,\lambda_1]} ch_n. \tag{24}$$