Class D and Symmetry in Fermionic Many-Body Systems

Ken Shiozaki

May 31, 2026

The class-D symmetry of a free-fermion system,

$$U_C \mathcal{H}^* U_C^{-1} = -\mathcal{H}, \quad U_C U_C^* = 1, \tag{1}$$

has two different counterparts as a symmetry of a quantum many-body system. This comes from the fact that, for a free-fermion Hamiltonian

$$H = \sum_{ij} \psi_i^\dagger \mathcal{H}_{ij} \psi_j, \tag{2}$$

the complex fermion $\psi_i^\dagger$ always has fermion-number symmetry $U(1)^f$:

$$e^{i\theta N} \psi_i^\dagger e^{-i\theta N} = e^{i\theta} \psi_i^\dagger. \tag{3}$$

1 $(-1)^F$ Symmetry

In general, a free-fermion Hamiltonian can be written in terms of Majorana fermions $\{a_i\}_{i=1,\dots,N}$ and an antisymmetric matrix $\mathcal{A}$ as

$$H = \sum_{ij} \frac{i}{2} \mathcal{A}_{ij} a_i a_j. \tag{4}$$

It has fermion-parity symmetry

$$(-1)^F a_i (-1)^F = -a_i. \tag{5}$$

Introducing complex fermions $\{c_i\}_{i=1,\dots,N}$ by

$$c_i = a_{2i-1} + i a_{2i}, \tag{6}$$

one can also write a general Hamiltonian with only fermion-parity symmetry as

$$H = \sum_{ij} c_i^\dagger h_{ij} c_j + \frac{1}{2} \sum_{ij} \Big(\Delta_{ij} c_i^\dagger c_j^\dagger + h.c. \Big) \tag{7}$$

$$\sim \frac{1}{2} \sum_{ij} \psi_i^\dagger \begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^T \end{pmatrix}_\tau \psi_j. \tag{8}$$

We have ignored the constant term and introduced the Nambu spinor

$$\psi_i = (c_i, c_i^\dagger)_\tau. \tag{9}$$

The BdG Hamiltonian

$$\mathcal{H}_{\mathrm{BdG}} = \begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^T \end{pmatrix}_\tau \tag{10}$$

has the following PHS by construction:

$$U_C \mathcal{H}^*_{\mathrm{BdG}} U_C^{-1} = -\mathcal{H}_{\mathrm{BdG}}, \quad U_C = \tau_x K. \tag{11}$$

This PHS is based on the redundancy of the internal degrees of freedom of the Nambu fermion: although ψ_i is written as a two-component spinor, it contains only one complex degree of freedom. This gives one interpretation of class D, namely $(-1)^F$ symmetry.

1.1 Zero-Dimensional Anomaly

The anomaly in zero spatial dimensions is classified by $\mathbb{Z}_2$ and is generated by a single Majorana fermion. The meaning of the $\mathbb{Z}_2$ classification is that two Majorana fermions a_1, a_2 can form a single state. Indeed, introducing

$$c = a_1 + i a_2, \tag{12}$$

the ground state of

$$H = +c^\dagger c \tag{13}$$

is the vacuum $|0\rangle$ of c.

2 $U(1)^f \rtimes \mathbb{Z}_2^C$ Symmetry

For complex fermions c_i, a general free-fermion Hamiltonian with $U(1)^f$ symmetry,

$$e^{i\theta N} c_i^\dagger e^{-i\theta N} = e^{i\theta} c_i^\dagger, \tag{14}$$

is written as

$$H = \sum_{ij} c_i^\dagger \mathcal{H}_{ij} c_j. \tag{15}$$

Now impose particle-hole symmetry in its original sense. Let $|0\rangle$ be the vacuum of the complex fermions c_i and define C by

$$C\psi_i C^{-1} = \psi_j^\dagger [U_C]_{ji}, \quad U_C U_C^* = 1, \quad C i C^{-1} = i, \quad C|0\rangle = |\mathrm{Full}\rangle. \tag{16}$$

The phase of the action of C on the vacuum can be set to 1 by a $U(1)^f$ transformation. Notice that C is unitary. Imposing C symmetry,

$$CHC^{-1} = H, \tag{17}$$

is equivalent, within free-fermion Hamiltonians and using Hermiticity of $\mathcal{H}$, to

$$U_C \mathcal{H}^* U_C^{-1} = -\mathcal{H}. \tag{18}$$

2.1 Zero-Dimensional Anomaly

The anomaly is again classified by $\mathbb{Z}_2$ and is generated by a single complex fermion. In the Hilbert space generated by one fermion, it is impossible to construct a single state preserving both $U(1)^f$ and $\mathbb{Z}_2^C$. Indeed, a linear combination of $c^\dagger|0\rangle$ and $|0\rangle$ is forbidden by $U(1)$ symmetry, while

$$Cc^\dagger|0\rangle = |0\rangle, \quad C|0\rangle = c^\dagger|0\rangle. \tag{19}$$

Therefore C symmetry forces a twofold degeneracy.

The $\mathbb{Z}_2$ classification can also be checked directly. Let c_1, c_2 be charge-one complex fermions, so that the $U(1)^f$ charge is $N = c_1^\dagger c_1 + c_2^\dagger c_2$, and impose PHS as

$$Cc_iC^{-1} = c_j^\dagger [U_C]_{ji}, \quad U_C^T = U_C, \quad C\left|0\right\rangle = c_1^\dagger c_2^\dagger \left|0\right\rangle . \tag{20}$$

Recall that any complex symmetric matrix A can be decomposed as $A = Q\Lambda Q^T$ with a unitary matrix Q. Thus, for a unitary matrix $V = (v_1, v_2)$, we can write

$$U_C = V^\dagger V^*. \tag{21}$$

Make a basis change of complex fermions that preserves the $U(1)^f$ symmetry and the canonical anticommutation relations:

$$c_i^\dagger = c_j'^\dagger V_{ji}. \tag{22}$$

Then PHS becomes

$$Cc_j'C^{-1} = c_j'^\dagger, \quad C\left|0\right\rangle = \det(V)c_1'^\dagger c_2'^\dagger \left|0\right\rangle . \tag{23}$$

Since

$$Cc_1'^\dagger \left|0\right\rangle = c_1' \det(V)c_1'^\dagger c_2'^\dagger \left|0\right\rangle = \det(V)c_2'^\dagger \left|0\right\rangle , \tag{24}$$

$$Cc_2'^\dagger \left|0\right\rangle = c_2' \det(V)c_1'^\dagger c_2'^\dagger \left|0\right\rangle = -\det(V)c_1'^\dagger \left|0\right\rangle , \tag{25}$$

the PHS action on the one-particle basis $c_1'^\dagger \left|0\right\rangle , c_2'^\dagger \left|0\right\rangle$ is

$$C(c_1'^\dagger \left|0\right\rangle , c_2'^\dagger \left|0\right\rangle) = (c_1'^\dagger \left|0\right\rangle , c_2'^\dagger \left|0\right\rangle) \begin{pmatrix} 0 & -\det(V) \\ \det(V) & 0 \end{pmatrix} . \tag{26}$$

Therefore the states

$$(c_1'^\dagger \pm ic_2'^\dagger) \left|0\right\rangle \tag{27}$$

preserve $U(1)^f \times \mathbb{Z}_2^C$ symmetry:

$$C(c_1'^\dagger \pm ic_2'^\dagger) \left|0\right\rangle = \pm i \det(V)(c_1'^\dagger \pm ic_2'^\dagger) \left|0\right\rangle . \tag{28}$$

2.2 Comment

Fermion-parity symmetry and $U(1)^f \rtimes \mathbb{Z}_2^C$ symmetry are different symmetries, so the classifications of invertible phases need not coincide. It is known from cobordism computations that the classifications differ in three spacetime dimensions. See Ref. [1].

References

[1] Luuk Stehouwer, *Interacting SPT phases are not Morita invariant*, arXiv:2110.07408.