Notes on the Chen–Gu–Liu–Wen Cocycle Model

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Abstract

I summarize the construction of the ground states and local unitary transformations of the Chen–Gu–Liu–Wen model [1], their properties in systems with boundary, and the symmetry action on boundary states.

1 Ground state on a sphere

Given a $(d+1)$-cocycle $\nu \in Z^{d+1}(G, U(1))$, the partition function that gives one SPT phase in spacetime dimension $(d+1)$ is

$$Z = \frac{1}{|G|^{N_v}} \sum_{\{g_x\}} \prod_{\Delta^{d+1}} \nu(g_0, g_1, \dots, g_d)^{\mathrm{sign}(\Delta^{d+1})} \tag{1}$$

where N_v is the number of vertices in spacetime. The factor $\mathrm{sign}(\Delta^p)$ assigns a sign ± 1 to the orientation of the p-simplex Δ^p. The ground state on the sphere S^d is obtained by performing the path integral over the point inside the sphere, which may be taken to be a single point:

$$|\nu\rangle = \frac{1}{|G|} \sum_{g_*} \frac{1}{|G|^{N_v}} \sum_{\{g_x\}} \prod_{\Delta} \nu(g_*, g_1, \dots, g_d)^{\mathrm{sign}(\Delta^d)} |\{g_x\}\rangle . \tag{2}$$

Here N_v is the number of vertices of the spatial sphere S^d. On the local basis $|g_x\rangle$, G acts as $\hat{g}\,|h_x\rangle = |gh_x\rangle$. Checking the G symmetry of the state $|\nu\rangle$, we have

$$\hat{g}\,|\nu\rangle = \frac{1}{|G|} \sum_{g_*} \frac{1}{|G|^{N_v}} \sum_{\{g_x\}} \prod_{\Delta} \nu(g_*, g_1, \dots, g_d)^{\mathrm{sign}(\Delta)} |\{gg_x\}\rangle \tag{3}$$

$$= \frac{1}{|G|} \sum_{g_*} \frac{1}{|G|^{N_v}} \sum_{\{g_x\}} \prod_{\Delta} \nu(g_*, g^{-1}g_1, \dots, g^{-1}g_d)^{\mathrm{sign}(\Delta)} |\{g_x\}\rangle \tag{4}$$

$$= \frac{1}{|G|} \sum_{g_*} \frac{1}{|G|^{N_v}} \sum_{\{g_x\}} \prod_{\Delta} \nu(g^{-1}g_*, g^{-1}g_1, \dots, g^{-1}g_d)^{\mathrm{sign}(\Delta)} |\{g_x\}\rangle . \tag{5}$$

Using the homogeneity condition for cochains,

$$\nu(gg_0, gg_1, \dots, gg_d) = \nu(g_0, g_1, \dots, g_d), \tag{6}$$

we confirm that

$$\hat{g}\,|\nu\rangle = |\nu\rangle . \tag{7}$$

In the expression for $|\nu\rangle$, the state does not depend on the group element g_* at the point inside the sphere. Indeed, applying the cocycle condition

$$(d\nu)(g_0, \dots, g_{d+1}) = \nu(g_1, g_2, \dots, g_{d+1})\nu(g_0, g_2, \dots, g_{d+1})^{-1} \cdots \nu(g_0, g_1, \dots, g_d)^{d+1} = 1 \tag{8}$$

to $(g_*, g'_*, g_1, \dots, g_d)$ gives

$$\nu(g_*, g_1, \dots, g_d)\nu(g'_*, g_1, \dots, g_d)^{-1} \tag{9}$$

$$= \nu(g_*, g'_*, g_2, g_3, \dots, g_d)\nu(g_*, g'_*, g_1, g_2, \dots, g_d)^{-1} \cdots \nu(g_*, g'_*, g_1, g_2, \dots, g_{d-1})^{d-1}. \tag{10}$$

The right-hand side cancels with contributions from adjacent d-simplices. Therefore, for any group element $g_* \in G$, we can write

$$|\nu\rangle = \frac{1}{|G|^{N_v}} \sum_{\{g_x\}} \prod_{\Delta^d} \nu(g_*, g_1, \dots, g_d)^{\text{sign}(\Delta^d)} |\{g_x\}\rangle . \tag{11}$$

Equivalently, the state is obtained by acting with the unitary transformation

$$U(\nu) = \sum_{\{g_x\}} \prod_{\Delta^d} \nu(g_*, g_1, \dots, g_d)^{\text{sign}(\Delta^d)} |\{g_x\}\rangle \langle\{g_x\}| \tag{12}$$

on

$$|\text{triv}\rangle = \bigotimes_x \frac{1}{|G|} \sum_{g_x \in G} |g_x\rangle . \tag{13}$$

Namely,

$$|\nu\rangle = U(\nu) |\text{triv}\rangle . \tag{14}$$

The G symmetry of the state $|\nu\rangle$ can be rephrased as

$$\hat{g} U(\nu) \hat{g}^{-1} = U(\nu). \tag{15}$$

One Hamiltonian whose ground state is $|\nu\rangle$ is given as follows. First, one Hamiltonian whose ground state is $|\text{triv}\rangle$ is

$$H_{\text{triv}} = -\sum_x P_x, \tag{16}$$

where P_x is the projection $P_x = |\phi_x\rangle \langle\phi_x|$ onto the trivial state $|\phi_x\rangle = \frac{1}{\sqrt{|G|}} \sum_{g_x \in G} |g_x\rangle$ at site x. A Hamiltonian whose ground state is $|\nu\rangle$ is then

$$H(\nu) = U(\nu) H_{\text{triv}} U(\nu)^{-1} = -\sum_x U(\nu) P_x U(\nu)^{-1}. \tag{17}$$

Notice that $U(\nu) P_x U(\nu)^{-1}$ has support only on sites adjacent to site x.

1.1 Example

Let us consider one example. Take one spatial dimension and $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{00, 01, 10, 11\}$. A nontrivial group cocycle is

$$\omega(g, h) = (-1)^{[g]_1 [h]_2}. \tag{18}$$

Here $[g]_j$ denotes the j-th component. Choosing $g_* = e$, we have

$$\nu(e, g_j, g_{j+1}) = \omega(g_j, g_j^{-1} g_{j+1}) = (-1)^{[g_j]_1 [-g_j + g_{j+1}]_2}. \tag{19}$$

Introduce Pauli matrices at the j-th site by $\tau^z |[g]_1[g]_2\rangle = (-1)^{[g]_1} |[g]_1[g]_2\rangle$ and $\sigma^z |[g]_1[g]_2\rangle = (-1)^{[g]_2} |[g]_1[g]_2\rangle$. Then

$$U(\nu) = \sum_{\{g_j\}} \prod_j \nu(e, g_j, g_{j+1}) |\{g_j\}\rangle \langle\{g_j\}| = \prod_j (-1)^{\frac{1-\tau_j^z}{2}\left(\frac{1-\sigma_j^z}{2} - \frac{1-\sigma_{j+1}^z}{2}\right)} = \prod_j e^{\frac{\pi i}{4}(1-\tau_j^z)(-\sigma_j^z + \sigma_{j+1}^z)}. \tag{20}$$

The trivial Hamiltonian can be given as a projection onto $\tau_j^x = \sigma_j^x = 1$, and if one follows the definition directly it is given by

$$H_0' = -\sum_j \frac{1+\tau_j^x}{2}\frac{1+\sigma_j^x}{2}. \tag{21}$$

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry operators are

$$\prod_j \tau_j^x, \quad \prod_j \sigma_j^x. \tag{22}$$

If the only requirement is to obtain the same ground state, another Hamiltonian may be used:

$$H_0 = -\sum_j (\tau_j^x + \sigma_j^x). \tag{23}$$

Below we use this expression. Noting that

$$U(\nu)\tau_j^x U(\nu)^{-1} = \sigma_j^z \tau_j^x \sigma_{j+1}^z, \tag{24}$$

$$U(\nu)\sigma_j^x U(\nu)^{-1} = \tau_{j-1}^z \sigma_j^x \tau_j^z, \tag{25}$$

we obtain the cluster Hamiltonian

$$H(\nu) = -\sum_j \sigma_j^z \tau_j^x \sigma_{j+1}^z - \sum_j \tau_{j-1}^z \sigma_j^x \tau_j^z. \tag{26}$$

2 Properties of the local unitary transformation $U(\nu)$ in the presence of a boundary

On the other hand, in a system with boundary, contributions from the right-hand side of (11) remain at the boundary, and therefore

- $U(\nu)$ depends on the choice of g_*, and
- equivalently, $U(\nu)$ is not G-invariant.

To make the calculation concrete, we confirm this point in spatial dimension $d = 1$. Consider a finite system with sites $x = 1, \ldots, N$.

Define the unitary transformation $U_{\text{open}}(\nu)$ simply by keeping only the terms in the expression for $U(\nu)$ whose support remains in the finite system:

$$U_{\text{open}}(\nu) = \sum_{g_1,\ldots,g_N} \prod_{x=1}^{N-1} \nu(g_*, g_x, g_{x+1}) \left|g_1, \ldots, g_N\right\rangle \left\langle g_1, \ldots, g_N\right| . \tag{27}$$

Because contributions from the two ends remain, $U_{\text{open}}(\nu)$ is not G-invariant. Indeed,

$$\hat{g} U_{\text{open}}(\nu)\hat{g}^{-1} = \sum_{g_1,\ldots,g_N} \prod_{x=1}^{N-1} \nu(g_*, g_x, g_{x+1}) \left|gg_1, \ldots, gg_N\right\rangle \left\langle gg_1, \ldots, gg_N\right| \tag{28}$$

$$= \sum_{g_1,\ldots,g_N} \prod_{x=1}^{N-1} \nu(g_*, g^{-1}g_x, g^{-1}g_{x+1}) \left|g_1, \ldots, g_N\right\rangle \left\langle g_1, \ldots, g_N\right| \tag{29}$$

$$= \sum_{g_1,\ldots,g_N} \prod_{x=1}^{N-1} \nu(gg_*, g_x, g_{x+1}) \left|g_1, \ldots, g_N\right\rangle \left\langle g_1, \ldots, g_N\right| \tag{30}$$

which corresponds to changing the reference group element as $g_* \mapsto gg_*$. Thus

$$= \sum_{g_1,\ldots,g_N} \prod_{x=1}^{N-1} \nu(g_*, g_x, g_{x+1})\nu(gg_*, g_*, g_x)\nu(gg_*, g_*, g_{x+1})^{-1} |g_1, \ldots, g_N\rangle \langle g_1, \ldots, g_N| \tag{31}$$

$$= \sum_{g_1,\ldots,g_N} \nu(gg_*, g_*, g_1) \left[\prod_{x=1}^{N-1} \nu(g_*, g_x, g_{x+1})\right] \nu(gg_*, g_*, g_N)^{-1} |g_1, \ldots, g_N\rangle \langle g_1, \ldots, g_N| \tag{32}$$

$$= \sum_{g'_1, g'_N} \sum_{g_1,\ldots,g_N} \nu(gg_*, g_*, g'_1) |g'_1\rangle \langle g'_1| \left[\prod_{x=1}^{N-1} \nu(g_*, g_x, g_{x+1})\right] |g_1, \ldots, g_N\rangle \langle g_1, \ldots, g_N| \nu(gg_*, g_*, g'_N)^{-1} |g'_N\rangle \langle g'_N| \tag{33}$$

$$= v_L U_{\text{open}}(\nu) v_R^\dagger, \tag{34}$$

and the G invariance is broken at the boundary. Here we introduced unitaries supported only on the boundary,

$$v_{L/R} = \nu(gg_*, g_*, g_{1/N}) |g_{1/N}\rangle \langle g_{1/N}| . \tag{35}$$

This does not mean that finite-dimensional degrees of freedom appear at the ends of the system and that $v_{L/R}$ acts as a projective representation.

As an example, let us apply $U_{\text{open}}(\nu)$ to the trivial Hamiltonian H_0 in the case $G = \mathbb{Z}_2 \times \mathbb{Z}_2$.

$$U_{\text{open}}(\nu)\tau_j^x U_{\text{open}}(\nu)^{-1} = \begin{cases} \sigma_1^z \tau_1^x \sigma_2^z & (j = 1), \\ \sigma_j^z \tau_j^x \sigma_{j+1}^z & (2 \le j \le N-1), \\ \tau_N^x & (j = N), \end{cases} \tag{36}$$

$$U_{\text{open}}(\nu)\sigma_j^x U_{\text{open}}(\nu)^{-1} = \begin{cases} \sigma_1^x \tau_1^z & (j = 1), \\ \tau_{j-1}^z \sigma_j^x \tau_j^z & (2 \le j \le N-1), \\ \tau_{N-1}^z \sigma_N^x & (j = N). \end{cases} \tag{37}$$

Therefore,

$$H_{\text{open}}(\nu) = U_{\text{open}}(\nu) H_0 U_{\text{open}}(\nu)^{-1} = -\sum_{j=1}^{N-1} \sigma_j^z \tau_j^x \sigma_{j+1}^z - \tau_N^x - \sigma_1^x \tau_1^z - \sum_{j=2}^{N-1} \tau_{j-1}^z \sigma_j^x \tau_j^z - \tau_{N-1}^z \sigma_N^x. \tag{38}$$

Thus the $\mathbb{Z}_2$ symmetry is broken at the ends. As a remark, since $H_{\text{open}}(\nu)$ is unitarily equivalent to H_0, its ground state is unique and gapped.

3 Symmetry action on boundary states

Decompose the Hamiltonian as

$$H = H_{\text{bulk}} + H_{\text{bdy}}. \tag{39}$$

Here $H_{\text{bulk}} = -\sum_j B_j, B_j = U(\nu) P_j U(\nu)^{-1}$, is the sum of local Hamiltonians at points in the interior of the system, and H_{bdy} is a sum of arbitrary local Hamiltonians at the boundary that respect the symmetry. Solving H_{bulk} leaves boundary degrees of freedom, and the ground-state manifold can be written as

$$\{|\Psi(\{g_n\})\rangle\}_{g_n \in G}. \tag{40}$$

Let g_n denote boundary degrees of freedom and g_j denote bulk degrees of freedom. The relative phases of the ground states $|\Psi(\{g_n\})\rangle$ are not fixed. We fix the phases of the ground states by

$$|\Psi(\{g_n\})\rangle = \sum_{\{g_j\}} \prod_{\Delta} \nu(g_*, g_0, \ldots, g_d)^{\text{sgn}\,\Delta} |\{g_j\}, \{g_n\}\rangle . \tag{41}$$

[1] The symmetry transformation $U(g \in G)$ is seen to close within the ground-state manifold, and its representation on the ground-state manifold can be calculated mechanically. The result of this calculation is shown in App. E of [2]; here I try to derive it.

3.1 One spatial dimension

First, let us practice the calculation in one spatial dimension. The ground-state wave function specified by the boundary spins g_1, g_N is

$$|\Psi(g_1, g_N)\rangle = \sum_{g_2,\ldots,g_{N-1}} \prod_{j=1}^{N-1} \nu(g_*, g_j, g_{j+1}) \, |g_1, g_2, \ldots, g_{N-1}, g_N\rangle \,. \tag{42}$$

The symmetry transformation is

$$U(g)\, |\Psi(g_1, g_N)\rangle \tag{43}$$
$$= \sum_{g_2,\ldots,g_{N-1}} \prod_{j=1}^{N-1} \nu(g_*, g_j, g_{j+1}) \, |gg_1, gg_2, \ldots, gg_{N-1}, gg_N\rangle \tag{44}$$
$$= \sum_{g_2,\ldots,g_{N-1}} \nu(g_*, g_1, g^{-1}g_2)\{\prod_{j=2}^{N-2} \nu(g_*, g^{-1}g_j, g^{-1}g_{j+1})\}\nu(g_*, g^{-1}g_{N-1}, g_N) \, |gg_1, g_2, \ldots, g_{N-1}, gg_N\rangle \tag{45}$$
$$= \sum_{g_2,\ldots,g_{N-1}} \nu(gg_*, gg_1, g_2)\{\prod_{j=2}^{N-2} \nu(gg_*, g_j, g_{j+1})\}\nu(gg_*, g_{N-1}, gg_N) \, |gg_1, g_2, \ldots, g_{N-1}, gg_N\rangle \,. \tag{46}$$

Using the cocycle condition

$$\nu(gg_*, g_j, g_{j+1}) = \nu(g_*, g_j, g_{j+1})\nu(g_*, gg_*, g_j)\nu(g_*, gg_*, g_{j+1})^{-1}, \tag{47}$$

we find

$$U(g)\, |\Psi(g_1, g_N)\rangle \tag{48}$$
$$= \sum_{g_2,\ldots,g_{N-1}} \nu(g_*, gg_1, g_2)\nu(g_*, gg_*, gg_1)\nu(g_*, gg_*, g_2)^{-1} \tag{49}$$
$$\nu(g_*, gg_*, g_2)\{\prod_{j=2}^{N-2} \nu(g_*, g_j, g_{j+1})\}\nu(g_*, gg_*, g_{N-1})^{-1} \tag{50}$$
$$\nu(g_*, g_{N-1}, gg_N)\nu(g_*, gg_*, g_{N-1})\nu(g_*, gg_*, gg_N)^{-1} \, |gg_1, g_2, \ldots, g_{N-1}, gg_N\rangle \tag{51}$$
$$= \sum_{g_2,\ldots,g_{N-1}} \nu(g_*, gg_*, gg_1)\nu(g_*, gg_1, g_2)\{\prod_{j=2}^{N-2} \nu(g_*, g_j, g_{j+1})\}\nu(g_*, g_{N-1}, gg_N)\nu(g_*, gg_*, gg_N)^{-1} \, |gg_1, g_2, \ldots, g_{N-1}, gg_N\rangle \tag{52}$$
$$= \nu(g_*, gg_*, gg_1)\nu(g_*, gg_*, gg_N)^{-1} \, |\Psi(gg_1, gg_N)\rangle \,. \tag{53}$$

Thus the action indeed closes on the ground-state manifold $\{|\Psi(g_1, g_N)\rangle\}_{g_1, g_N \in G}$. Moreover, note that the phase factors $\nu(g_*, gg_*, gg_1)$ and $\nu(g_*, gg_*, gg_N)^{-1}$ are written only in terms of the degrees of freedom at the left and right ends, respectively. Following [2], define operators on the ground-state manifold $|\Psi(g_1, g_N)\rangle$ by

$$S(g)\, |\Psi(g_1, g_N)\rangle = |\Psi(gg_1, gg_N)\rangle \,, \tag{54}$$
$$N(g)\, |\Psi(g_1, g_N)\rangle = \nu(g_*, gg_*, g_1)\nu(g_*, gg_*, g_N) \, |\Psi(g_1, g_N)\rangle \,. \tag{55}$$

[1] In [3], ghost spins are introduced outside the system to fix the relative phases, but they are unnecessary here. Note that the constructions of the ground state in [3] and [1] are different.

Then, on the ground-state manifold, the symmetry action has the representation

$$U(g) = N(g)S(g). \tag{56}$$

This representation separates into the degrees of freedom at the left and right ends. Writing the edge degrees of freedom as $|g_1\rangle$ and $|g_N\rangle$, we have

$$U_1(g) = N_1(g)S_1(g), \quad U_1(g)\,|g_1\rangle = \nu(g_*, gg_*, g_1)\,|gg_1\rangle\,, \tag{57}$$

$$U_N(g) = N_N(g)S_N(g), \quad U_N(g)\,|g_N\rangle = \nu(g_*, gg_*, g_N)^{-1}\,|gg_N\rangle\,. \tag{58}$$

Furthermore, setting $g_* = e$, we have $\nu(e, g, gh) = \omega(g, h)$, and hence

$$U_1(g)\,|g_1\rangle = \omega(g, g_1)\,|gg_1\rangle\,, \quad U_N(g)\,|g_N\rangle = \omega(g, g_N)^{-1}\,|gg_N\rangle\,. \tag{59}$$

These are nothing but the regular representations determined by the two 2-cocycles ω and ω^{-1}, respectively.

3.2 Two spatial dimensions

Next, we calculate the case of two spatial dimensions. Let the sign $\operatorname{sgn}\Delta$ of a 2-simplex be $+1$ when 012 is counterclockwise and -1 when 012 is clockwise. Set $g_* = e$ from the outset. The ground-state wave function specified by boundary spins g_n is

$$|\Psi(\{g_n\})\rangle = \sum_{\{g_j\}} \prod_{\Delta} \nu(e, g_0, g_1, g_2)^{\operatorname{sgn}\Delta}\,|\{g_j\}, \{g_n\}\rangle\,. \tag{60}$$

The symmetry transformation is

$$U(g)\,|\Psi(\{g_n\})\rangle \tag{61}$$

$$= \sum_{\{g_j\}} \prod_{\Delta} \nu(e, g_0, g_1, g_2)^{\operatorname{sgn}\Delta}\,|\{gg_j\}, \{gg_n\}\rangle\,. \tag{62}$$

For contributions from bulk simplices,

$$\sum_{\{g_j\}} \cdots \nu(e, g_0, g_1, g_2)^{\operatorname{sgn}\Delta} \cdots |\cdots gg_0, gg_1, gg_2 \cdots\rangle \tag{63}$$

$$= \sum_{\{g_j\}} \cdots \nu(e, g^{-1}g_0, g^{-1}g_1, g^{-1}g_2)^{\operatorname{sgn}\Delta} \cdots |\cdots g_0, g_1, g_2 \cdots\rangle \tag{64}$$

$$= \sum_{\{g_j\}} \cdots \nu(g, g_0, g_1, g_2)^{\operatorname{sgn}\Delta} \cdots |\cdots g_0, g_1, g_2 \cdots\rangle\,. \tag{65}$$

If the same transformation is performed for simplices that include the boundary, then, for example with $0, 1$ boundary degrees of freedom, we have

$$\sum_{\{g_j\}} \cdots \nu(e, g_0, g_1, g_2)^{\operatorname{sgn}\Delta} \cdots |\cdots gg_0, gg_1, g_2 \cdots\rangle \tag{66}$$

$$= \sum_{\{g_j\}} \cdots \nu(e, g_0, g_1, g^{-1}g_2)^{\operatorname{sgn}\Delta} \cdots |\cdots gg_0, gg_1, g_2 \cdots\rangle \tag{67}$$

$$= \sum_{\{g_j\}} \cdots \nu(g, gg_0, gg_1, g_2)^{\operatorname{sgn}\Delta} \cdots |\cdots gg_0, gg_1, g_2 \cdots\rangle\,, \tag{68}$$

and similarly for other cases. As a result,

$$U(g)\,|\Psi(\{g_n\})\rangle = \sum_{\{g_j\}} \prod_{\Delta} \nu(g, \tilde{g}_0, \tilde{g}_1, \tilde{g}_2)^{\operatorname{sgn}\Delta}\,|\{g_j\}, \{gg_n\}\rangle\,, \tag{69}$$

where

$$\tilde{g}_x = \begin{cases} g_x & x \in \text{ bulk}, \\ g g_x & x \in \text{ bdy}, \end{cases} \tag{70}$$

By the cocycle condition,

$$\nu(g, g_0, g_1, g_2) = \nu(e, g_0, g_1, g_2)\nu(e, g, g_1, g_2)^{-1}\nu(e, g, g_0, g_2)\nu(e, g, g_0, g_1)^{-1}. \tag{71}$$

Using this, we obtain

$$U(g) \left|\Psi(\{g_n\})\right\rangle = \sum_{\{g_j\}} \prod_{\Delta} [\nu(e, \tilde{g}_0, \tilde{g}_1, \tilde{g}_2)\nu(e, g, \tilde{g}_1, \tilde{g}_2)^{-1}\nu(e, g, \tilde{g}_0, \tilde{g}_2)\nu(e, g, \tilde{g}_0, \tilde{g}_1)^{-1}]^{\text{sgn}\,\Delta} \left|\{g_j\}, \{g g_n\}\right\rangle. \tag{72}$$

In the bulk, the factors $\nu(e, g, g_1, g_2)^{-1}\nu(e, g, g_0, g_2)\nu(e, g, g_0, g_1)^{-1}$ cancel, but at the boundary the factors remain, giving

$$U(g) \left|\Psi(\{g_n\})\right\rangle = \prod_n \nu(e, g, g g_n, g g_{n+1})^{-\text{sgn}\,\Delta^1_{n,n+1}} \left|\Psi(\{g g_n\})\right\rangle. \tag{73}$$

Here the boundary sites are labeled counterclockwise, and the sign $\text{sgn}\,\Delta^1_{n,n+1} \in \{\pm 1\}$ is defined to be $+1$ if the orientation of the 1-simplex $(n, n+1)$ is counterclockwise and -1 if it is clockwise. Notice that it depends on the branching structure. On the ground-state manifold, introduce

$$S(g) \left|\Psi(\{g_n\})\right\rangle = \left|\Psi(\{g g_n\})\right\rangle, \tag{74}$$

$$N(g) \left|\Psi(\{g_n\})\right\rangle = \prod_n \nu(e, g, g_n, g_{n+1})^{-\text{sgn}\,\Delta^1_{n,n+1}} \left|\Psi(\{g_n\})\right\rangle = \prod_n \omega(g, g^{-1}g_n, g_n^{-1}g_{n+1})^{-\text{sgn}\,\Delta^1_{n,n+1}} \left|\Psi(\{g_n\})\right\rangle. \tag{75}$$

Then the symmetry action on the ground-state manifold is given by

$$U(g) = N(g)S(g). \tag{76}$$

References

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[3] Michael Levin, Zheng-Cheng Gu, *Braiding statistics approach to symmetry-protected topological phases*, arXiv:1202.3120.