

# Cohomology Groups of $BSO$ and $BO$

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June 1, 2026

## Abstract

The integral cohomology of  $BSO$  and  $BO$  was completely determined in Refs. [2, 3]. Following the notation of Ref. [2], I check the low-dimensional cohomology groups. I also review low-dimensional spin bordism groups.

## 1 Integral cohomology of $BO_n$

The integral cohomology of  $BSO_n$  and  $BO_n$  is completely determined in Refs. [2, 3]. I do not follow the proof here, but only check the result of Ref. [2]. For  $G_n = U_n, O_n, SO_n$ , let  $E_n \rightarrow BG_n$  be the universal principal  $G_n$ -bundle, and define the universal  $n$ -plane bundle by

$$\zeta(G_n) \begin{cases} E_n \times_{G_n} \mathbb{R}^n & (G_n = SO_n, O_n), \\ E_n \times_{G_n} \mathbb{C}^n & (G_n = U_n). \end{cases} \quad (1.1)$$

Here

$$E_n \times_{G_n} \mathbb{R}^n = E_n \times \mathbb{R}^n / (e, v) \sim (eg, g^{-1}v). \quad (1.2)$$

Let  $\rho$  and  $\delta$  be the homomorphisms in the following long exact sequence:

$$\rightarrow H^q(X; \mathbb{Z}) \xrightarrow{2} H^q(X; \mathbb{Z}) \xrightarrow{\rho} H^q(X; \mathbb{Z}_2) \xrightarrow{\delta} H^{q+1}(X; \mathbb{Z}) \rightarrow . \quad (1.3)$$

As a remark, by the universal coefficient theorem, once the integral cohomology is known, the homology is also known:

$$H^i(X, \mathbb{Z}) = \text{Hom}(H_i(X), \mathbb{Z}) \oplus \text{Ext}(H_{i-1}(X), \mathbb{Z}) = \text{Free } H_i(X) \oplus \text{Tor } H_{i-1}(X). \quad (1.4)$$

**Theorem 1.1** ([2]). *Set*

$$p_q = (-1)^q c_{2q}(\zeta(O_n) \otimes \mathbb{C}) \in H^{4q}(BO_n; \mathbb{Z}), \quad (1.5)$$

where  $\zeta(O_n) \otimes \mathbb{C}$  is the complexification of  $\zeta(O_n)$ . Then  $H^*(BO_n; \mathbb{Z}) \cong \mathcal{R}_n/\mathcal{I}_n$ , where

$$\mathcal{R}_n = \mathbb{Z}[p_1, \dots, p_{\lfloor n/2 \rfloor}, \delta(w_1^\epsilon w_{2i_1} w_{2i_2} \cdots w_{2i_l}) \mid \epsilon \in \{0, 1\}, 0 < i_1 < \cdots < i_l \leq \lfloor n/2 \rfloor]. \quad (1.6)$$

Here  $\mathcal{I}_n$  is the ideal generated by the following relations:

(i)  $2\delta(w_1^\epsilon w_{2i_1} \cdots w_{2i_l}) = 0$ .

(ii) If  $n$  is even,  $\delta(w_1 w_n) = 0$ .

(iii) If  $n$  is even,  $(\delta w_n)^2 = p_{n/2} \delta w_1$ .

(iv) For a set  $I = \{\epsilon/2, i_1, \dots, i_s\}$ , write  $w(2I) = w_1^\epsilon w_{2i_1} \cdots w_{2i_s}$  and  $p(I) = (\delta w_1)^\epsilon p_{i_1} \cdots p_{i_s}$ . Then

$$\delta w(2I) \delta w(2J) = \sum_{k \in I} \delta w_{2k} \cdot p\left((I - \{k\}) \cap J\right) \cdot \delta w\left(2\left(\left((I - \{k\}) \cup J\right) - \left((I - \{k\}) \cap J\right)\right)\right). \quad (1.7)$$

Moreover,

$$\rho(p_q) = w_{2q}^2, \quad \rho\delta = Sq^1, \quad (1.8)$$

and under Whitney sum,

$$p_q \mapsto \sum r_{2q-j} \otimes r_j, \quad \text{where } r_{2s} = p_s, \quad r_{2s+1} = (\delta w_{2s})^2 + p_s \delta w_1. \quad (1.9)$$

Although it is not written explicitly in Ref. [2], if one sets

$$w(2\emptyset) = 0, \quad \delta w(2\emptyset) = 0, \quad (1.10)$$

$$p(\emptyset) = 1, \quad (1.11)$$

then both sides of relation (iv) have the same degree. When  $I = \{k\}$ , condition (iv) is an identity, so it is meaningful only when  $|I| > 1$ . Thus one may assume  $|I| > 1$  and  $|J| \geq 1$ . Therefore relation (iv) may be ignored at least below degree 6, the degree of  $\delta(w_1 w_2) \delta(w_1)$ .<sup>1</sup> Listing the low-degree generators gives

$$\text{deg} \begin{array}{c|cccccccccc} \delta w_1 & \delta w_2 & \delta(w_1 w_2) & p_1 & \delta w_4 & \delta(w_1 w_4) & \delta(w_2 w_4) & \delta(w_1 w_2 w_4) & p_2 & \cdots \\ \hline 2 & 3 & 4 & 4 & 5 & 6 & 7 & 8 & 8 & \cdots \end{array} \quad (1.13)$$

Thus, for sufficiently large  $n$ ,

$$H^1(BO_n; \mathbb{Z}) = 0, \quad (1.14)$$

$$H^2(BO_n; \mathbb{Z}) = \mathbb{Z}_2[\delta w_1], \quad (1.15)$$

$$H^3(BO_n; \mathbb{Z}) = \mathbb{Z}_2[\delta w_2], \quad (1.16)$$

$$H^4(BO_n; \mathbb{Z}) = \mathbb{Z}[p_1] \oplus \mathbb{Z}_2[(\delta w_1)^2] \oplus \mathbb{Z}_2[\delta(w_1 w_2)], \quad (1.17)$$

$$H^5(BO_n; \mathbb{Z}) = \mathbb{Z}_2[\delta w_4] \oplus \mathbb{Z}_2[\delta w_1 \delta w_2]. \quad (1.18)$$

<sup>1</sup>In fact, in the case  $I = \{1/2, 1\}$  and  $J = \{1/2\}$ , relation (iv) gives

$$\delta(w_1 w_2) \delta(w_1) = \delta w_1 p(\emptyset) \delta w(2\{1/2, 1\}) + \delta w_2 p(\{1/2\}) \delta w(2\emptyset) = \delta w_1 \delta(w_1 w_2), \quad (1.12)$$

which is a trivial relation.

## 2 Spin bordism groups

Remark: for the third spin bordism group  $\Omega_3^{\text{spin}}(X)$ , a computation method is known from Ref. [4].

The  $E^2$  page of the AHSS converging to the spin bordism group

$$\Omega_d^{\text{spin}}(X) \quad (2.1)$$

is

$$E_{p,q}^2 = H_p(X, \Omega_q^{\text{spin}}). \quad (2.2)$$

It has the form

$q = 4$	$\mathbb{Z}$	$H_1(X)$	$H_2(X)$	$H_3(X)$	$H_4(X)$	$H_5(X)$	(2.3)
$q = 3$	$0$	$0$	$0$	$0$	$0$	$0$	
$q = 2$	$\mathbb{Z}_2$	$H_1(X, \mathbb{Z}_2)$	$H_2(X, \mathbb{Z}_2)$	$H_3(X, \mathbb{Z}_2)$	$H_4(X, \mathbb{Z}_2)$	$H_5(X, \mathbb{Z}_2)$	
$q = 1$	$\mathbb{Z}_2$	$H_1(X, \mathbb{Z}_2)$	$H_2(X, \mathbb{Z}_2)$	$H_3(X, \mathbb{Z}_2)$	$H_4(X, \mathbb{Z}_2)$	$H_5(X, \mathbb{Z}_2)$	
$q = 0$	$\mathbb{Z}$	$H_1(X)$	$H_2(X)$	$H_3(X)$	$H_4(X)$	$H_5(X)$	
	$p = 0$	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	

## 3 Backup

The known cohomology rings of  $BO$  are

$$H^*(BO, \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, w_3, \dots], \quad (3.1)$$

$$H^*(BO, K) = K[p_1, p_2, p_3, \dots], \quad (3.2)$$

where  $K$  is a field of characteristic not equal to 2,  $w_i$  are the Stiefel–Whitney classes, and  $p_i$  are the Pontryagin classes. By Theorem 16.17 of Switzer [1],

$$H_*(BO, \mathbb{Z}_2) = \mathbb{Z}_2[x_1, x_2, \dots]. \quad (3.3)$$

The universal coefficient theorem says

$$H_i(X, A) = H_i(X) \otimes A \oplus \text{Tor}(H_{i-1}(X), A), \quad (3.4)$$

$$H^i(X, A) = \text{Hom}(H_i(X), A) \oplus \text{Ext}(H_{i-1}(X), A). \quad (3.5)$$

We use

$\text{Tor}(A, B)$	$B = \mathbb{Z}$	$B = \mathbb{Z}_n$		$\text{Ext}(A, B)$	$B = \mathbb{Z}$	$B = \mathbb{Z}_n$	(3.6)
$A = \mathbb{Z}$	$0$	$0$		$A = \mathbb{Z}$	$0$	$0$	
$A = \mathbb{Z}_m$	$0$	$\mathbb{Z}_{\text{gcd}(m,n)}$		$A = \mathbb{Z}_m$	$\mathbb{Z}_m$	$\mathbb{Z}_{\text{gcd}(m,n)}$	

In particular,

$$H^i(X, \mathbb{Z}) = \text{Hom}(H_i(X), \mathbb{Z}) \oplus \text{Ext}(H_{i-1}(X), \mathbb{Z}) = \text{Free } H_i(X) \oplus \text{Tor } H_{i-1}(X). \quad (3.7)$$

Also,

$$H^*(BO, \mathbb{Z}) \otimes \mathbb{R} = H^*(BO, \mathbb{R}) = \mathbb{R}[p_1, p_2, \dots], \quad (3.8)$$

so the degrees in which free  $\mathbb{Z}$  summands occur in  $H^*(BO, \mathbb{Z})$  are known; only the torsion part is the issue. Start from

$$H_1(BO) = \pi_1(BO)/[\pi_1(BO), \pi_1(BO)] = \mathbb{Z}_2. \quad (3.9)$$

Let  $p \neq 2$  be a prime. Looking in low degrees at

$$H^i(BO, \mathbb{Z}_p) = \text{Hom}(H_i(BO), \mathbb{Z}_p) \oplus \text{Ext}(H_{i-1}(BO), \mathbb{Z}_p), \quad (3.10)$$

we find

$$0 = \text{Hom}(H_2(BO), \mathbb{Z}_p) \oplus \text{Ext}(H_1(BO), \mathbb{Z}_p), \quad (3.11)$$

$$0 = \text{Hom}(H_3(BO), \mathbb{Z}_p) \oplus \text{Ext}(H_2(BO), \mathbb{Z}_p), \quad (3.12)$$

$$\mathbb{Z}_p = \text{Hom}(\underbrace{H_4(BO)}_{\cong \mathbb{Z}}, \mathbb{Z}_p) \oplus \text{Ext}(H_3(BO), \mathbb{Z}_p). \quad (3.13)$$

On the other hand,

$$\text{Hom}(\mathbb{Z}, \mathbb{Z}_p) = \mathbb{Z}_p, \quad \text{Hom}(\mathbb{Z}_n, \mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & (p|n), \\ 0 & \text{else,} \end{cases} \quad (3.14)$$

so  $H_2(BO), H_3(BO), H_4(BO)$  can only contain finite sums of groups  $\mathbb{Z}_{2^q}$ , apart from the known free part.

Now look at low degrees in

$$H_i(BO, \mathbb{Z}_2) = H_i(BO) \otimes \mathbb{Z}_2 \oplus \text{Tor}(H_{i-1}(BO), \mathbb{Z}_2). \quad (3.15)$$

First,

$$\mathbb{Z}_2^2 = H_2(BO, \mathbb{Z}_2) = H_2(BO) \otimes \mathbb{Z}_2 \oplus \underbrace{\text{Tor}(H_1(BO), \mathbb{Z}_2)}_{\mathbb{Z}_2}. \quad (3.16)$$

Using

$$\mathbb{Z}_m \otimes \mathbb{Z}_n = \mathbb{Z}_{\text{gcd}(m,n)}, \quad (3.17)$$

we see that

$$H_2(BO) = \mathbb{Z}_{2^p}, \quad p \geq 1. \quad (3.18)$$

Similarly,

$$\mathbb{Z}_2^3 = H_3(BO, \mathbb{Z}_2) = H_3(BO) \otimes \mathbb{Z}_2 \oplus \underbrace{\text{Tor}(H_2(BO), \mathbb{Z}_2)}_{\mathbb{Z}_2} \quad (3.19)$$

implies

$$H_3(BO) = \mathbb{Z}_{2^q} \oplus \mathbb{Z}_{2^r}, \quad q, r \geq 1, \quad (3.20)$$

and

$$\mathbb{Z}_2^5 = H_4(BO, \mathbb{Z}_2) = H_4(BO) \otimes \mathbb{Z}_2 \oplus \underbrace{\text{Tor}(H_3(BO), \mathbb{Z}_2)}_{\mathbb{Z}_2^2} \quad (3.21)$$

implies

$$H_4(BO) = \mathbb{Z} \oplus \mathbb{Z}_{2^s} \oplus \mathbb{Z}_{2^t}, \quad s, t \geq 1. \quad (3.22)$$

I would like to determine the powers of 2, but this is left unresolved here.

## References

- [1] R. M. Switzer, *Algebraic Topology—Homotopy and Homology*.
- [2] Edgar H. Brown, *The cohomology of  $BSO_n$  and  $BO_n$  with integer coefficients*, Proceedings of the American Mathematical Society 85:2, 283–288.
- [3] Mark Feshbach, *The Integral Cohomology Rings of the Classifying Spaces of  $O(n)$  and  $SO(n)$* , Indiana Univ. Math. J. 32 No. 4 (1983), 511–516.
- [4] Greg Brumfiel and John Morgan, *The Pontrjagin Dual of 3-Dimensional Spin Bordism*, arXiv:1612.02860.