Computing Homotopy Classes of Loops in SO(n)

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1 Preparation

The rotation matrix in the xy plane is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = e^{\theta L}, \quad L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

More generally, a rotation by $\theta_{[ij]}$ in the $x_i x_j$ plane of $\mathbb{R}^n$ is given by the $SO(n)$ matrix

$$e^{\theta_{[ij]} L_{[ij]}}, \quad [L_{[ij]}]_{kl} = -\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}. \tag{2}$$

The symbol $[ij]$ denotes an antisymmetric component. In n dimensions there are $n(n-1)/2$ such components. Any $SO(n)$ matrix R can be written as

$$R = e^{\sum_{[ij]} \theta_{[ij]} L_{[ij]}}. \tag{3}$$

To determine the $n(n-1)/2$ parameters $\{\theta_{[ij]}\}_{[ij]}$ from a given $SO(n)$ matrix, proceed as follows. First diagonalize R:

$$R = U \Lambda U^\dagger, \quad \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n), \quad \lambda_j \in U(1). \tag{4}$$

The eigenvalues have the following structure, depending on the parity of n:

$$\{e^{i\alpha_1}, e^{-i\alpha_1}, \dots, e^{i\alpha_{n/2}}, e^{-i\alpha_{n/2}}\}, \quad (n \text{ even}), \tag{5}$$

$$\{e^{i\alpha_1}, e^{-i\alpha_1}, \dots, e^{i\alpha_{(n-1)/2}}, e^{-i\alpha_{(n-1)/2}}, 1\}, \quad (n \text{ odd}). \tag{6}$$

For each eigenvalue λ_j, choose the argument $\phi_j = \mathrm{Arg}(\lambda_j)$ so that

$$\boxed{-\pi < \phi_j < \pi. \tag{7}}$$

Then

$$R(t) = U \,\mathrm{diag}(e^{it\phi_1}, \dots, e^{it\phi_n}) U^\dagger \tag{8}$$

is regarded as the "shortest path" connecting $R(0) = 1_n$ to $R(1) = R$. If an eigenvalue with $\mathrm{Arg}(\lambda_j) = \pi$ is present, this shortest path is not uniquely determined, so such cases are excluded.

Solving the linear equation

$$\left.\frac{d}{dt} R(t)\right|_{t=0} = U \,\mathrm{diag}(i\phi_1, \dots, i\phi_n) U^\dagger = \sum_{[ij]} \theta_{[ij]} L_{[ij]} \tag{9}$$

gives the parameters $\{\theta_{[ij]}\}_{[ij]}$.

- One still needs to prove that the parameters $\{\theta_{[ij]}\}_{[ij]}$ obtained by this method are unique.

2 Computing the First Homotopy Class

Using $\{\theta_{[ij]}\}_{[ij]}$, fix a lift

$$SO(n) \to Spin(n). \tag{10}$$

The generators of $Spin(n)$ with the same structure constants as $\{L_{[ij]}\}_{[ij]}$ are given, in terms of gamma matrices $\{\gamma_i\}_{i=1,\ldots,n}$ satisfying

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}, \tag{11}$$

by

$$\Sigma_{[ij]} = \frac{[\gamma_i, \gamma_j]}{-4}. \tag{12}$$

Define the lift by

$$e^{\sum_{[ij]} \theta_{[ij]} L_{[ij]}} \mapsto e^{\sum_{[ij]} \theta_{[ij]} \Sigma_{[ij]}}. \tag{13}$$

For a sequence of points $(R_1, \ldots, R_N)$ in $SO(n)$, we compute the first homotopy class of the loop obtained by connecting $R_1 \to R_2 \to \cdots \to R_N \to R_1$ with the shortest paths defined above. Set

$$R^{p\to p+1} = R_{p+1} R_p^{-1}. \tag{14}$$

If $R^{p\to p+1}$ has -1 as an eigenvalue, the shortest path is not unique, so the sequence $(R_1, \ldots, R_N)$ is excluded. Following the method above, compute the N sets of parameters $\{\theta_{[ij]}^{p\to p+1}\}_{[ij]}$ defined by

$$R^{p\to p+1} = e^{\sum_{[ij]} \theta_{[ij]}^{p\to p+1} \Sigma_{[ij]}}. \tag{15}$$

Then compute the $Spin(n)$ matrix

$$q := e^{\sum_{[ij]} \theta_{[ij]}^{N\to 1} \Sigma_{[ij]}} \cdots e^{\sum_{[ij]} \theta_{[ij]}^{2\to 3} \Sigma_{[ij]}} e^{\sum_{[ij]} \theta_{[ij]}^{1\to 2} \Sigma_{[ij]}}. \tag{16}$$

By construction, $q \in \{\mathbf{1}, -\mathbf{1}\}$.

- If $q = \mathbf{1}$, the homotopy class is trivial.
- If $q = -\mathbf{1}$, the homotopy class is nontrivial.

3 Example: $SO(4)$

Take

$$\{L_{[12]}, L_{[23]}, L_{[31]}, L_{[14]}, L_{[24]}, L_{[34]}\} \tag{17}$$

to be the standard antisymmetric generators in the corresponding coordinate planes. For example, choose gamma matrices

$$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} = \{\sigma_x\tau_x, \sigma_y\tau_x, \sigma_z\tau_x, \tau_y\}. \tag{18}$$

Then

$$\Sigma_{[ij]} = \frac{[\gamma_i, \gamma_j]}{-4}, \tag{19}$$

$$(\Sigma_{[12]}, \Sigma_{[23]}, \Sigma_{[31]}, \Sigma_{[14]}, \Sigma_{[24]}, \Sigma_{[34]}) = -\frac{i}{2}(\sigma_z, \sigma_x, \sigma_y, \sigma_x\tau_z, \sigma_y\tau_z, \sigma_z\tau_z), \tag{20}$$

and the $L_{[ij]}$ and $\Sigma_{[ij]}$ have the same structure constants.

Consider the following sequence of points in $SO(4)$:

$$
\begin{aligned}
R_1 &= 1_4, && (21)\\
R_2 &= e^{\frac{\pi}{2}L_{[12]}+\frac{\pi}{2}L_{[34]}}, && (22)\\
R_3 &= e^{\pi L_{[12]}+\pi L_{[34]}} = -I_4, && (23)\\
R_4 &= e^{-\frac{\pi}{2}L_{[12]}+\frac{\pi}{2}L_{[34]}}. && (24)
\end{aligned}
$$

This is block diagonal and corresponds, in the xy plane, to the rotation $0 \to \pi/2 \to \pi \to 3\pi/2 \to 2\pi$, so it belongs to the nontrivial homotopy class.

By definition, compute $\{\theta_{[ij]}^{p\to p+1}\}_{[ij]}$. One finds

$$
\begin{aligned}
R^{1\to 2}, R^{2\to 3}: &\quad \theta_{[ij]} = \left(\frac{\pi}{2}, 0, 0, 0, 0, \frac{\pi}{2}\right), && (25)\\
R^{3\to 4}, R^{4\to 1}: &\quad \theta_{[ij]} = \left(\frac{\pi}{2}, 0, 0, 0, 0, -\frac{\pi}{2}\right). && (26)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
q^{1\to 2} &= \exp\left[-\frac{\pi i}{4}\sigma_z - \frac{\pi i}{4}\sigma_z\tau_z\right], && (27)\\
q^{2\to 3} &= \exp\left[-\frac{\pi i}{4}\sigma_z - \frac{\pi i}{4}\sigma_z\tau_z\right], && (28)\\
q^{3\to 4} &= \exp\left[-\frac{\pi i}{4}\sigma_z + \frac{\pi i}{4}\sigma_z\tau_z\right], && (29)\\
q^{4\to 1} &= \exp\left[-\frac{\pi i}{4}\sigma_z + \frac{\pi i}{4}\sigma_z\tau_z\right]. && (30)
\end{aligned}
$$

Multiplying them gives

$$
q^{4\to 1}q^{3\to 4}q^{2\to 3}q^{1\to 2} = -1_4. \tag{31}
$$