C^*-geometric phase: Viennot–Lages formalism (draft)

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I summarize the formulation of geometric phases in open quantum systems introduced in [1]. Some examples are also computed in [2].

Let $\mathcal{H}_\mathcal{S}$ and $\mathcal{H}_\mathcal{E}$ be the Hilbert spaces of the system and the environment, respectively. The Hilbert space of the composite system consisting of the system and the environment is $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$. We write the inner product on $\mathcal{H}_\mathcal{S}$ as $\langle .|.\rangle_\mathcal{S}$, the inner product on $\mathcal{H}_\mathcal{E}$ as $\langle .|.\rangle_\mathcal{E}$, and the inner product of the total system as $\langle\langle .|.\rangle\rangle$. We also write the wave function of the total system as ψ, or as $|\psi\rangle\rangle$. We denote the Hamiltonians of the system and the environment by $H_\mathcal{S}$ and $H_\mathcal{E}$, respectively, and the interaction between the system and the environment by H_I. The Hamiltonian of the total system is

$$H = H_\mathcal{S} \otimes 1_\mathcal{E} + 1_\mathcal{S} \otimes H_\mathcal{E} + H_I. \tag{1}$$

We write $\mathcal{D}(\mathcal{H}_\mathcal{S}) = \{\rho \in \mathcal{L}(\mathcal{H}_\mathcal{S}) | \rho^\dagger = \rho, \rho \geq 0, \mathrm{tr}\,_\mathcal{S}\rho = 1\}$. For a wave function ψ of the total system, the density matrix ρ of the system is

$$\rho = \mathrm{Tr}\,_\mathcal{E}|\psi\rangle\rangle\langle\langle\psi|. \tag{2}$$

In the Viennot–Lages formulation, the Hilbert space $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$ of the total system is regarded as a C^*-module over the operator algebra $\mathcal{L}(\mathcal{H}_\mathcal{S})$ of the system. The C^*-module $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$ is equipped with the following inner product:

$$\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E} \times \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E} \longrightarrow \mathcal{L}(\mathcal{H}_\mathcal{S}), \tag{3}$$

$$(|\psi\rangle\rangle, |\phi\rangle\rangle) \mapsto \langle\langle\psi|\phi\rangle\rangle_* := \mathrm{Tr}\,_\mathcal{E}|\phi\rangle\rangle\langle\langle\psi|. \tag{4}$$

More explicitly, if we write a basis of $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$ as $|n\rangle \otimes |a\rangle$ and write

$$\phi = \phi_{na} |n\rangle \otimes |a\rangle\,, \tag{5}$$

$$\psi = \psi_{na} |n\rangle \otimes |a\rangle\,, \tag{6}$$

then

$$\langle\phi|\psi\rangle_* = \mathrm{Tr}\,_\mathcal{E} \sum_{namb} \phi^*_{na}\psi_{mb} \mathrm{Tr}\,_\mathcal{E}[|m\rangle \otimes |b\rangle \langle n| \otimes \langle a|] = \sum_{nm}\sum_a \phi^*_{na}\psi_{ma} |m\rangle \langle n|\,. \tag{7}$$

This inner product has the following properties:

- It is linear from the right and anti-linear from the left. Namely, for $A, B \in \mathcal{L}(\mathcal{H}_\mathcal{S})$ and $\psi, \phi, \chi \in \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$,

$$\langle\psi|A\phi + B\chi\rangle_* = \mathrm{Tr}\,_\mathcal{E}[(A|\phi\rangle\rangle + B|\chi\rangle\rangle)\langle\langle\psi|] \tag{8}$$

$$= A\,\langle\psi|\phi\rangle_* + B\,\langle\psi|\chi\rangle_*\,, \tag{9}$$

$$\langle A\psi + B\chi|\phi\rangle_* = \mathrm{Tr}\,_\mathcal{E}[(|\phi\rangle\rangle(\langle\langle\psi|A^\dagger + \langle\langle\chi|B^\dagger)] \tag{10}$$

$$= \langle\psi|\phi\rangle_*\,A^\dagger + \langle\chi|\phi\rangle_*\,B^\dagger. \tag{11}$$

- It is "Hermitian." Namely, for $\psi, \phi \in \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$,

$$\langle\phi|\psi\rangle_*^\dagger = (\sum_{nm}\sum_a \phi_{na}^*\psi_{ma} |m\rangle\langle n|)^\dagger \tag{12}$$
$$= \sum_{nm}\sum_a \phi_{na}\psi_{ma}^* |n\rangle\langle m| \tag{13}$$
$$= \langle\psi|\phi\rangle_* \tag{14}$$

- It is positive definite. Namely, for $\psi \in \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$,

$$\langle\psi|\psi\rangle_* = \sum_{nm}\sum_a \psi_{na}^*\psi_{ma} |m\rangle\langle n| \tag{15}$$

  is $\psi\psi^\dagger$ if $\psi = (\psi)_{na}$ is regarded as a rectangular matrix, and therefore it is positive semidefinite:

$$\langle\psi|\psi\rangle_* \geq 0. \tag{16}$$

  Moreover, $\psi\psi^\dagger = O$ means that all singular values of the rectangular matrix ψ are zero, so $\langle\psi|\psi\rangle_* = 0$ implies $\psi = 0$.

When the state $\psi \in \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$ of the total system is normalized, the density matrix ρ of the system is given as the $*$-norm by

$$\rho = \|\psi\|_*^2 = \langle\psi|\psi\rangle_*. \tag{17}$$

Definition 1 (Eigenoperator and $*$-eigenvector [1])**.** *Let H be the Hamiltonian of the total system. A pair $E \in \mathcal{L}(\mathcal{H}_\mathcal{S}), \phi_E \in \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$ satisfying the following conditions is called an eigenoperator of H and a mixed eigenstate whose eigenoperator is E, respectively:*

$$[E \otimes 1_\mathcal{E}, H] = 0, \tag{18}$$
$$H\phi_E = E\phi_E. \tag{19}$$

In other words, the stationary Schrodinger equation of the total system is regarded as an eigenvalue problem as an $\mathcal{L}(\mathcal{H}_\mathcal{S})$-module, and a constraint is imposed on the eigenvalue E as an operator of the system. When $E = \lambda 1_\mathcal{S}$, this is just the ordinary eigenvalue equation itself.

Some properties of the eigenoperator E and the mixed eigenstate ϕ_E are summarized in the appendix of [1].

(i) (Property 9 in [1]) Let $\mathcal{L}$ be the Lindblad operator. The eigenoperator E and the mixed eigenstate ρ_E satisfy

$$\mathcal{L}(\rho_E) = E\rho_E - \rho_E E^\dagger, \tag{20}$$

$$\mathcal{L}(E\rho_E - \rho_E E^\dagger) = E\mathcal{L}(\rho_E) - \mathcal{L}(\rho_E)E^\dagger. \tag{21}$$

In [1], it is claimed from this property that the eigenoperator E can be found if the Lindblad operator $\mathcal{L}$ is given, even when the Hamiltonian H of the total system is unknown.

(ii) (Property 10 in [1]) E is almost Hermitian in the following sense:

$$\mathrm{tr}_{\mathcal{S}}[\rho_E(E - E^\dagger)] = 0. \tag{22}$$

(iii) (Proposition 11 in [1]) For any E_0 satisfying $[H, E_0 \otimes 1_{\mathcal{E}}] = 0$, let $\lambda \in \mathbb{C}$ and $\phi_{E,\lambda}$ be a solution of the ordinary eigenvalue equation

$$H\phi_{E,\lambda} = (E_0 + \lambda)\phi_{E,\lambda}. \tag{23}$$

Then, for the eigenoperator $E = E_0 + \lambda$, $\phi_{E,\lambda}$ is a mixed eigenstate.

Proof. (i) follows from

$$\mathcal{L}(\rho_E) = \mathrm{tr}_{\mathcal{E}}[H, |\phi_E\rangle\rangle\langle\langle\phi_E|] = \mathrm{tr}_{\mathcal{E}}[E\phi_E\rangle\rangle\langle\langle\phi_E| - |\phi_E\rangle\rangle\langle\langle\phi_E|E^\dagger] = E\rho_E - \rho_E E^\dagger, \tag{24}$$

and

$$\begin{aligned}
\mathcal{L}(E\rho_E - \rho_E E^\dagger) &= \mathrm{tr}_{\mathcal{E}}([H, E|\phi_E\rangle\rangle\langle\langle\phi_E|] - [H, |\phi_E\rangle\rangle\langle\langle\phi_E|E^\dagger]) &(25)\\
&= \mathrm{tr}_{\mathcal{E}}(HE|\phi_E\rangle\rangle\langle\langle\phi_E| - E|\phi_E\rangle\rangle\langle\langle\phi_E|H - H|\phi_E\rangle\rangle\langle\langle\phi_E|E^\dagger + |\phi_E\rangle\rangle\langle\langle\phi_E|E^\dagger H) &(26)\\
&= \mathrm{tr}_{\mathcal{E}}(EH|\phi_E\rangle\rangle\langle\langle\phi_E| - E|\phi_E\rangle\rangle\langle\langle\phi_E|H - H|\phi_E\rangle\rangle\langle\langle\phi_E|E^\dagger + |\phi_E\rangle\rangle\langle\langle\phi_E|HE^\dagger) &(27)\\
&= E\mathcal{L}(\rho_E) - \mathcal{L}(\rho_E)E^\dagger. &(28)
\end{aligned}$$

(ii) follows from

$$\begin{aligned}
\mathrm{tr}_{\mathcal{S}}[\rho_E(E - E^\dagger)] &= \mathrm{tr}_{\mathcal{S}}\mathrm{tr}_{\mathcal{E}}(|\phi_E\rangle\rangle\langle\langle\phi_E|(E - E^\dagger)) &(29)\\
&= \langle\langle\phi_E|(E - E^\dagger)|\phi_E\rangle\rangle &(30)\\
&= \langle\langle\phi_E|(H - H^\dagger)|\phi_E\rangle\rangle &(31)\\
&= 0. &(32)
\end{aligned}$$

(iii) is obvious. □

Fix E and examine the ambiguity of the eigenvector ϕ_E. Let K be the unitary transformation group of the environment and the symmetry group that keeps the Hamiltonian of the total system invariant:

$$K = \{k \in \mathcal{U}(\mathcal{H}_{\mathcal{E}}) | (1_{\mathcal{S}} \otimes k)H(1_{\mathcal{S}} \otimes k^\dagger) = H\}. \tag{33}$$

Let G_E be the largest subgroup of invertible operators of the system that keeps the eigenspace with eigenoperator E invariant. Namely, G_E is defined as the largest subgroup $G_E \subset \mathcal{GL}(\mathcal{H}_{\mathcal{S}})$ such that

$$G_E \ker(H - E \otimes 1_{\mathcal{E}}) = \{g\phi \in \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}} | g \in \mathcal{GL}(\mathcal{H}_{\mathcal{S}}), H\phi = E\phi\} \subset \ker(H - E \otimes 1_{\mathcal{E}}) \tag{34}$$

holds. When $E = \lambda 1_{\mathcal{S}}$, G_E is nothing but the subgroup of invertible operators that keeps the eigenspace invariant in the presence of degeneracy. Choosing $\phi_E \in \ker(H - E \otimes 1_{\mathcal{E}})$ is nothing but choosing one vector from the degenerate eigenspace. The group $G_E \times K$ can be regarded as the ambiguity of the eigenvector ϕ_E. Namely, when $gk \in G_E \times K$,

$$(H - E \otimes 1_{\mathcal{E}})\phi_E = 0 \quad \Rightarrow \quad (H - E \otimes 1_{\mathcal{E}})gk\phi_E = k(H - E \otimes 1_{\mathcal{E}})g\phi_E = 0. \tag{35}$$

The transformation $\phi_E \mapsto gk\phi_E$ induces the transformation of the density matrix

$$\rho_E = \langle \phi_E | \phi_E \rangle_* \quad \Rightarrow \quad \mathrm{Tr}_{\mathcal{E}}[gk|\phi_E\rangle\rangle\langle\langle\phi_E|g^\dagger k^\dagger] = g\rho_E g^\dagger. \tag{36}$$

Notice that, because g is not unitary, the property $\mathrm{tr}_{\mathcal{S}}[\rho_E] = 1$ is not preserved.

Furthermore, introduce a subgroup $J_E^0(\rho_E) \subset G_E$ as follows. For the density matrix $\rho_E = \langle \phi_E | \phi_E \rangle_*$, define the G_E-orbit by

$$G_E \rho_E := \{g\rho_E g^\dagger | g \in G_E\}. \tag{37}$$

We define $J_E^0(\rho_E)$ to be the group obtained by collecting the stabilizers at all points $g\rho_E g^\dagger \in G_E\rho_E$ of the orbit. Namely,

$$J_E^0(\rho_E) := \{j \in G_E | \exists g \in G_E, jg\rho_E g^\dagger j^\dagger = g\rho_E g^\dagger\}. \tag{38}$$

Since

$$j\rho_E j^\dagger = \rho_E \quad \Leftrightarrow \quad gjg^{-1}g\rho_E g^\dagger (gjg^{-1})^\dagger = g\rho_E g^\dagger \tag{39}$$

holds, $J_E^0(\rho_E)$ itself is also the G_E-orbit starting from the stabilizer at ρ_E:

$$J_E^0(\rho_E) = \bigcup_{g \in G_E} g\{j \in G_E | j\rho_E j^\dagger = \rho_E\}g^{-1}. \tag{40}$$

Proposition 1 (Property 11 in [1])**.** *$J_E^0(\rho_E)$ is a normal subgroup of G_E.* [a]

[a] In the general theory of 2-gauge groups, the image of the inclusion $t : J \to G$ must be a normal subgroup of G.

Proof. When $j \in J_E^0(\rho_E)$, there exists some $g \in G_E$ such that $jg\rho_E g^\dagger j^\dagger = g\rho_E g^\dagger$. If $h \in G_E$, then $hjh^{-1}hg\rho_E(hg)^\dagger(hjh^{-1})^\dagger = hjg\rho_E g^\dagger j^\dagger h^\dagger = hg\rho_E(hg)^\dagger$, and therefore $hjh^{-1} \in J_E^0(\rho_E)$. □

Remark: In [1], when ρ_E is not full rank, J^1 is introduced and it is claimed that its Lie algebra is solvable (Property 12), but Eq. (A.10) is incorrect. Since there is no need to introduce J^1 at this stage, I do not introduce J^1 for now.

So far we have considered a single pair H, E. From now on, suppose that H and E depend on points of a parameter space M, and consider the situation where a smooth eigenoperator $E(x) \in \mathcal{L}(\mathcal{H}_{\mathcal{S}})$ over M is given such that

$$[H(x), E(x) \otimes 1_{\mathcal{E}}] = 0, \quad x \in M. \tag{41}$$

For each patch U^α of M, we give an eigenvector

$$H(x)\phi_E(x) = E(x)\phi_E(x), \quad x \in U^\alpha \tag{42}$$

as a "gauge fixing." Then the patch transformation of the density matrix $\rho_E^\alpha(x) = \langle \phi_E^\alpha(x) | \phi_E^\alpha(x) \rangle_*$ is given by

$$\rho_E^\alpha(x) = g^{\alpha\beta}(x)\rho_E^\beta(x)g^{\alpha\beta}(x)^\dagger, \quad g^{\alpha\beta}(x) \in G_{E(x)}, \quad x \in U^\alpha \cap U^\beta. \tag{43}$$

It follows from

$$\rho_E^\alpha(x) = g^{\alpha\beta}(x)g^{\beta\gamma}(x)g^{\gamma\alpha}(x)\rho_E^\alpha(x)(g^{\alpha\beta}(x)g^{\beta\gamma}(x)g^{\gamma\alpha}(x))^\dagger, \quad x \in U^\alpha \cap U^\beta \cap U^\gamma \tag{44}$$

that the product $g^{\alpha\beta}(x)g^{\beta\gamma}(x)g^{\gamma\alpha}(x)$ belongs to the stabilizer $\{g \in G_{E(x)} | g\rho_E^\alpha(x)g^\dagger = \rho_E^\alpha(x)\}$ of $\rho_E^\alpha(x)$. The definition of $J_{E(x)}^0(\rho_E(x))$ is obtained by collecting the stabilizers of density matrices that are gauge equivalent, $\rho_E(x) \sim g\rho_E(x)g^\dagger$, so there is no need to attach the patch label, and hence

$$h^{\alpha\beta\gamma}(x) := g^{\alpha\beta}(x)g^{\beta\gamma}(x)g^{\gamma\alpha}(x) \in J_{E(x)}^0(\rho_E(x)). \tag{45}$$

Therefore, the cocycle condition is not satisfied and the structure of 2-gauge theory appears; this is the claim of [1].

Introduce a connection. For a locally given $\phi_E(x)$, define an $\mathcal{L}(\mathcal{H}_\mathcal{S})$-valued 1-form A so that the following condition is satisfied:

$$\mathcal{A}\|\phi_E\|_*^2 = \langle\phi_E|d\phi_E\rangle_* \,. \tag{46}$$

In [1], this definition is said to generalize the definition of the Berry connection for non-Hermitian systems with gain and loss due to the environment. Noting that $\rho_E = \|\phi_E\|_*^2$, $\mathcal{A}$ satisfies

$$d\rho_E = \mathcal{A}\rho_E + \rho_E\mathcal{A}^\dagger. \tag{47}$$

Proof.

$$\begin{aligned} d\rho_E &= d\,\langle\phi_E|\phi_E\rangle_* && (48)\\ &= \langle d\phi_E|\phi_E\rangle_* + \langle\phi_E|d\phi_E\rangle_* && (49)\\ &= \mathcal{A}\|\phi_E\|_*^2 + \|\phi_E\|_*^2\mathcal{A}^\dagger && (50)\\ &= \mathcal{A}\rho_E + \rho_E\mathcal{A}^\dagger. && (51)\end{aligned}$$

□

Under the gauge transformation

$$\tilde{\phi}_E = gk\phi_E, \quad g\in G_E, k\in K, \tag{52}$$

the connection $\mathcal{A}$ transforms as follows:

$$\begin{aligned} \tilde{\mathcal{A}}\|\tilde{\phi}_E\|^2 &= \langle\tilde{\phi}_E|d\tilde{\phi}_E\rangle_* && (53)\\ &= \mathrm{Tr}_{\mathcal{E}}[(dgk|\phi_E\rangle\rangle + gdk|\phi_E\rangle\rangle + gk|d\phi_E\rangle\rangle)\langle\langle\phi_E|g^\dagger k^\dagger] && (54)\\ &= dg\rho_E g^\dagger + g\,\langle\phi_E|k^\dagger dk|\phi_E\rangle_*\, g^\dagger + g\mathcal{A}\rho_E g^\dagger. && (55)\end{aligned}$$

Noting that $\|\tilde{\phi}_E\|^2 = g\rho_E g^\dagger$,

$$g^{-1}\tilde{\mathcal{A}}g\rho_E = \mathcal{A}\rho_E + g^{-1}dg\rho_E + \langle\phi_E|k^\dagger dk|\phi_E\rangle_* \,. \tag{56}$$

Here, introduce η as a solution of

$$\eta\|\phi_E\|^2 = \langle\phi_E|k^\dagger dk|\phi_E\rangle_* \,. \tag{57}$$

Then

$$g^{-1}\tilde{\mathcal{A}}g\rho_E = (\mathcal{A} + g^{-1}dg + \eta)\rho_E. \tag{58}$$

In [1], although it is noted that ρ_E is not invertible in general, the next step concludes that

$$g^{-1}\tilde{\mathcal{A}}g = \mathcal{A} + g^{-1}dg + \eta. \tag{59}$$

This, however, is incorrect because it holds only when ρ_E is invertible. In App. B of [1], the gauge transformation when ρ_E is not invertible is discussed in relation to the gauge group J^1. The term η represents the deviation from an ordinary gauge transformation and is characteristic of 2-gauge theory. Checking the possible values of η, we find that

$$\eta\rho_E + \rho_E\eta = \langle\phi_E|k^\dagger dk|\phi_E\rangle_* + \langle\phi_E|k^\dagger dk|\phi_E\rangle_*^\dagger = \langle\phi_E|k^\dagger dk + dk^\dagger k|\phi_E\rangle_* = 0 \tag{60}$$

and hence η belongs to the Lie algebra of the stabilizer of ρ_E (a subgroup of $J_E^0(\rho_E)$). In other words, it is the Lie algebra to which the failure of the cocycle condition belongs. Also note from the construction that η arises from a gauge transformation of the environment.

References

[1] David Viennot, Jose Lages, *A new kind of geometric phases in open quantum systems and higher gauge theory*, arXiv:1101.2852.

[2] David Viennot, Jose Lages, *C^*-geometric phase for mixed states: entanglement, decoherence and spin system*, arXiv:1101.2852.