A Note on Delicate Insulators Protected by Crystalline Symmetry

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May 30, 2026

Abstract

This is a note on a simple example of a delicate insulator protected by crystalline symmetry, explained to me by Nelson-san [1].

Consider a two-dimensional system with translation symmetry and mirror symmetry:

$$(x,y)\mapsto(x+1,y),\quad (x,y)\mapsto(x,y+1),\quad R:(x,y)\mapsto(-x,y). \tag{1}$$

We consider a 2×2 model; equivalently, there are two internal degrees of freedom in a unit cell. The Hamiltonian satisfies

$$RH(k_x,k_y)R^{-1}=H(-k_x,k_y),\quad R^2=1. \tag{2}$$

Here we assume that $R(k_x)$ is independent of k_x. We also assume that a band gap is present.

Suppose moreover that the representations of R carried by the occupied and empty states at $k_x=0,\pi$ are

	$k_x=0$	$k_x=\pi$
empty	$-$	$-$
occupied	$+$	$+$.

(3)

This holds for every k_y on the mirror-invariant lines. Therefore, in the basis

$$R=\begin{pmatrix}1 & \\ & -1\end{pmatrix}, \tag{4}$$

the Bloch state at $k_x=0,\pi$ is independent of k_y and can be written as

$$u(k_x\in\{0,\pi\},k_y)=\begin{pmatrix}1\\0\end{pmatrix}. \tag{5}$$

Define the Wilson line in the k_y direction from the occupied state by

$$e^{i\gamma(k_x)}=e^{-\oint dk_y\,\langle u(k_x,k_y)|\partial_{k_y}u(k_x,k_y)\rangle}. \tag{6}$$

Since the state is independent of k_y at $k_x=0,\pi$,

$$e^{i\gamma(k_x=0)}=e^{i\gamma(k_x=\pi)}=1. \tag{7}$$

Thus one can define the winding number of the Berry phase $\gamma(k_x)$ along $k_x=0\to\pi$, giving a $\mathbb{Z}$ classification.

The conclusion is unchanged if one adds any number of $R=-$ states to the empty states. However, if an $R=+$ state is added to the empty states, then at $k_x=0,\pi$ the Bloch state can take the form

$$u(k_x\in\{0,\pi\},k_y)=\begin{pmatrix}*\\*\\0\end{pmatrix}, \tag{8}$$

and hence an arbitrary Berry phase $\gamma(k_x\in\{0,\pi\})$ can appear. The $\mathbb{Z}$ classification is then destroyed.

References

[1] Aleksandra Nelson, Titus Neupert, Tomas Bzdusek, and A. Alexandradinata, *Multicellularity of delicate topological insulators*, arXiv:2009.01863.