Detectability Lemma

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Abstract

Following [1], I record a proof of the detectability lemma. It seems that [2] contains a simpler proof, but I have not checked it.

I first describe the setup. Consider the Hilbert space $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$ on a one-dimensional lattice. Here n is the number of sites. The local terms H_i of the Hamiltonian $H = \sum_i H_i$ are assumed to have support only on nearest-neighbor pairs of sites. For any finite-range Hamiltonian, this assumption can always be arranged in this form. Assume that the lowest eigenvalue of H_i is 0, and that this lowest eigenvalue is nondegenerate. Then $H \geq 0$. Indeed, for an arbitrary state $|\psi\rangle$,

$$\langle \psi \,|\, H \,|\, \psi \rangle = \sum_i \langle \psi \,|\, H_i \,|\, \psi \rangle \,, \tag{1}$$

and $\langle \psi \,|\, H_i \,|\, \psi \rangle \geq 0$. **The ground state need not be unique, but if there is degeneracy, the corresponding energy eigenvalues are assumed to be exactly degenerate.** Let $\mathcal{H}_0$ denote the ground-state subspace, and let $\mathcal{H}'$ denote its orthogonal complement. Let the spectral gap of H be $\epsilon > 0$. That is, assume

$$|\psi\rangle \in \mathcal{H}' \Rightarrow \langle \psi \,|\, H \,|\, \psi \rangle \geq \epsilon > 0. \tag{2}$$

The Hamiltonian is assumed to be frustration free. Namely,

$$|\Omega\rangle \in \mathcal{H}_0 \Leftrightarrow {}^{\forall} i,\ H_i \,|\Omega\rangle = 0 \tag{3}$$

is assumed to hold. Define the orthogonal projection obtained by "flattening" the local Hamiltonian H_i as the operator in which the lowest eigenvalue of H_i is changed to 0 and all the other eigenvalues are changed to 1. Concretely, if

$$H_i = \sum_{E_n^{(i)} > 0} E_n^{(i)} \,|n^{(i)}\rangle \langle n^{(i)}| \tag{4}$$

is a diagonalization, then take

$$Q_i = \sum_{E_n^{(i)} > 0} |n^{(i)}\rangle \langle n^{(i)}| \,. \tag{5}$$

This deformation preserves the frustration-free condition:

$$|\Omega\rangle \in \mathcal{H}_0 \Leftrightarrow {}^{\forall} i,\ Q_i \,|\Omega\rangle = 0. \tag{6}$$

Introduce the projection

$$P_i = 1 - Q_i = |0^{(i)}\rangle \langle 0^{(i)}| \,. \tag{7}$$

Then

$$|\Omega\rangle \in \mathcal{H}_0 \Leftrightarrow {}^{\forall} i,\ P_i \,|\Omega\rangle = |\Omega\rangle \,. \tag{8}$$

Introduce the following operator, defined as a product of projections onto local ground states:

$$A := \Pi_{\text{even}}\Pi_{\text{odd}}, \tag{9}$$
$$\Pi_{\text{even}} = P_2 P_4 P_6 \cdots, \tag{10}$$
$$\Pi_{\text{odd}} = P_1 P_3 P_5 \cdots. \tag{11}$$

Here note that neighboring projections P_{2i-1}, P_{2i} and P_{2i}, P_{2i+1} need not commute, whereas all other pairs commute. Also note that

$$|\Omega\rangle \in \mathcal{H}_0 \Rightarrow A|\Omega\rangle = A^\dagger|\Omega\rangle = |\Omega\rangle. \tag{12}$$

Moreover,

$$|\psi\rangle \in \mathcal{H}' \Rightarrow A|\psi\rangle \in \mathcal{H}' \tag{13}$$

holds. Indeed, since $\langle\Omega\,|\,A\,|\,\psi\rangle = \langle\Omega\,|\,\psi\rangle = 0$, the ground-state component of $A|\psi\rangle$ is zero. Thus, with respect to the decomposition $\mathcal{H}_0 \oplus \mathcal{H}'$, the operator A has the form

$$A = \begin{pmatrix} 1 & \\ & A|_{\mathcal{H}'} \end{pmatrix}. \tag{14}$$

Because A preserves the ground states and suppresses the excited-state component, one expects that, for sufficiently large l, A^l is close to the projection onto the ground-state subspace. Let

$$H_Q = \sum_i Q_i \tag{15}$$

be the Hamiltonian obtained by flattening the local terms. Assume that this deformation leaves the ground-state subspace unchanged and also keeps the spectral gap finite[1]. Let the spectral gap of H_Q be ϵ_Q. The detectability lemma states the following:

$$||A|_{\mathcal{H}'}|| < \frac{1}{(\epsilon_Q/2+1)^{\frac{1}{3}}}. \tag{16}$$

In other words, if the spectral gap is finite, then the action of A on $\mathcal{H}'$ is strictly smaller than 1.

(Proof) Take a normalized state $|\psi\rangle \in \mathcal{H}'$ from the orthogonal complement. Set $|\phi\rangle := A|\psi\rangle$. Since $|\phi\rangle \in \mathcal{H}'$, we have

$$\langle\phi\,|\,H_Q\,|\,\phi\rangle \geq \epsilon_Q ||\phi||^2. \tag{17}$$

We estimate $\langle\phi\,|\,H_Q\,|\,\phi\rangle$ from above. First note that, since $Q_{i\in\text{odd}}\Pi_{\text{odd}} = 0$, odd i do not contribute. Rewrite A as

$$A = \underbrace{(P_1P_3P_2)}_{\Delta_1}\underbrace{(P_5P_7P_6)}_{\Delta_2}\cdots\underbrace{(P_4P_8\cdots)}_{R} =: \Delta_1\Delta_2\cdots\Delta_m R. \tag{18}$$

Here $m \sim n/4$. Note that $R^\dagger = R$. Estimate $\langle\phi\,|\,Q_{4i-2}\,|\,\phi\rangle$:

$$\langle\phi\,|\,Q_{4i-2}\,|\,\phi\rangle = ||(1-P_{4i-2})A\psi|| = ||(1-P_{4i-2})\Delta_1\Delta_2\cdots R\psi|| \tag{19}$$
$$= ||\Delta_1\cdots\Delta_{i-1}(1-P_{4i-2})\Delta_i\cdots R\psi|| \leq ||(1-P_{4i-2})\Delta_i\cdots R\psi||. \tag{20}$$

Recall that $||Av|| \leq ||A||||v||$[2]. Set

$$v_i = \Delta_i\Delta_{i+1}\cdots R|\psi\rangle, \qquad v_{m+1} = R|\psi\rangle. \tag{21}$$

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[1] This has not been shown here.

[2] This follows from $||A|| := \max_{v\neq 0}||Av||/||v|| \geq ||Av||/||v||$.

Note that

$$||v_i|| = ||\Delta_i v_{i+1}|| \leq ||\Delta_i||\,||v_{i+1}|| \leq ||v_{i+1}||. \tag{22}$$

We have the expression

$$\langle \phi \,|\, Q_{4i-2} \,|\, \phi \rangle \leq ||(1 - P_{4i-2})\Delta_i v_{i+1}||^2 = ||(1 - P_{4i-2})P_{4i-3}P_{4i-1}P_{4i-2}v_{i+1}||^2, \qquad v_i = \Delta_i v_{i+1}. \tag{23}$$

Then, by (39) with $X = P_{4i-3}P_{4i-1}$, $Y = P_{4i-2}$, and $v = v_{i+1}/||v_{i+1}||$, we obtain

$$||(1 - P_{4i-2})\Delta_i v_{i+1}||^2 \leq \left(1 - \frac{||v_i||^2}{||v_{i+1}||^2}\right) \frac{||v_i||^2}{||v_{i+1}||^2} ||v_{i+1}||^2 \leq \left(1 - \frac{||v_i||^2}{||v_{i+1}||^2}\right). \tag{24}$$

Here we used $||v_i|| \leq ||v_{m+1}|| = ||R\,|\psi\rangle\,|| \leq ||R|| = 1$. Define

$$a_i = \frac{||v_i||^2}{||v_{i+1}||^2}, \quad i = 1, \ldots, m, \qquad a_{m+1} = ||R\psi||^2, \tag{25}$$

$$a_1 \cdots a_{m+1} = ||v_1||^2 = ||\phi||^2. \tag{26}$$

Then we have obtained

$$\langle \phi \,|\, Q_{4i-2} \,|\, \phi \rangle \leq 1 - a_i. \tag{27}$$

Thus the following estimate follows:

$$\langle \phi | \sum_i Q_{4i-2} | \phi \rangle = \sum_{i=2,6,\cdots} (1 - a_i), \qquad a_1 \cdots a_{m+1} = ||\phi||^2. \tag{28}$$

Using the method of Lagrange multipliers to find the extremum[3], the maximum is attained when $a_i = ||\phi||^{2/(m+1)}$. Therefore

$$\langle \phi | \sum_i Q_{4i-2} | \phi \rangle \leq m(1 - ||\phi||^{\frac{2}{m}}). \tag{29}$$

We want an estimate independent of m. Expanding with $||\phi||^2 = 1 + \delta x$ gives

$$m(1 - (1 + \delta x)^{1/m}) \sim -\delta x + \frac{1}{2}\frac{1}{m}\left(1 - \frac{1}{m}\right)\delta x^2. \tag{30}$$

Since

$$\frac{1}{m}\left(1 - \frac{1}{m}\right) = \frac{1}{4} - \left(\frac{1}{2} - \frac{1}{m}\right)^2, \tag{31}$$

it should be enough to bound this from above by a function of the form

$$-\delta x + c\delta x^2, \qquad c \geq \frac{1}{8}. \tag{32}$$

One can show that

$$m(1 - x^{\frac{1}{m}}) \leq \frac{1 - x}{\sqrt{x}}, \qquad x \in [0, 1] \tag{33}$$

[4]. Hence

$$\langle \phi | \sum_i Q_{4i-2} | \phi \rangle \leq \frac{1 - ||\phi||^2}{||\phi||}. \tag{34}$$

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[3] Since the expression is symmetric, it is probably obvious that the extremum occurs at $a_i = \text{const}$.

[4] Let $f_m(x) = m\sqrt{x}(1 - x^{1/m}) - (1 - x)$. Then $f_m''(x) = \frac{(m^2-4)x^{\frac{1}{m}} - m^2}{4mx^{\frac{3}{2}}} \geq 0$ is shown. Thus $f_m'(x)$ is monotonically decreasing on $[0, 1]$, and since $f_m'(1) = 0$, one obtains $f_m'(x \in [0, 1]) \geq 0$. Therefore $f_m(x)$ is monotonically increasing on $[0, 1]$, and since $f_m(1) = 0$, $f_m(x \in [0, 1]) \leq 0$.

Similarly, for Q_{4i}, for example by redefining $A = P_2 P_1 (P_4 P_6 \cdots)(P_3 P_5 \cdots)$, one obtains in the same way

$$\langle\phi| \sum_{i=4,8,\ldots} Q_{4i} |\phi\rangle \leq \frac{1 - ||\phi||^2}{||\phi||} \tag{35}$$

[5].

Combining these estimates gives

$$\epsilon_Q ||\phi||^2 \leq \langle\phi|H_Q|\phi\rangle \leq 2\frac{1 - ||\phi||^2}{||\phi||}. \tag{36}$$

It follows that

$$\epsilon_Q ||\phi||^3 \leq 2(1 - ||\phi||^2) \leq 2(1 - ||\phi||^3), \tag{37}$$

and therefore

$$||\phi|| \leq \frac{1}{(1 + \epsilon_Q/2)^{\frac{1}{3}}}. \tag{38}$$

This proves the claim. □

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[5] For a finite number of sites, one should carefully consider what happens at the right boundary.

A Lemma

We prove the following lemma. Let $\mathcal{H}$ be a Hilbert space, and let X, Y be orthogonal projections. Let $v \in \mathcal{H}$ be a vector of norm 1. Assume that $||XYv|| = 1 - \epsilon$. Then

$$||(1-Y)XYv||^2 \leq \epsilon(1-\epsilon) \tag{39}$$

holds.

(Proof) First note that, for any orthogonal projection P,

$$||(1-P)v||^2 + ||Pv||^2 = ((1-P)v, (1-P)v) + (Pv, Pv) = (v, (1-P)v) + (v, Pv) = (v, v) = ||v||^2. \tag{40}$$

What we want to show is

$$||(1-Y)XYv||^2 = ||XYv||^2 - ||YXYv||^2 = 1 - \epsilon - ||YXYv||^2 \leq \epsilon(1-\epsilon). \tag{41}$$

Thus it suffices to show

$$||YXYv|| \geq 1 - \epsilon. \tag{42}$$

This follows from

$$||XYv||^2 = (XYv, XYv) = (v, YXYv) \leq ||v|| \cdot ||YXYv|| = ||YXYv||. \tag{43}$$

This proves the lemma. □

References

[1] Dorit Aharonov, Itai Arad, Zeph Landau, Umesh Vazirani, *Quantum Hamiltonian complexity and the detectability lemma*, arXiv:1011.3445.

[2] Anurag Anshu, Itai Arad, Thomas Vidick, *A simple proof of the detectability lemma and spectral gap amplification*, arXiv:1602.01210.