The Eckart–Young–Mirsky Theorem

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Abstract

This is a memo on the proof of the Eckart–Young–Mirsky theorem, which gives the best rank-k approximation of a matrix.

Let $A \in \mathrm{Mat}_{m,n}(\mathbb{C})$ be an $m \times n$ matrix, and write its singular value decomposition as

$$A = U\Sigma V^\dagger = \sum_{i=1}^{\mathrm{rank}(A)} \sigma_i u_i v_i^\dagger, \quad \sigma_1 \geq \sigma_2 \geq \dots. \tag{1}$$

The Frobenius norm is defined by

$$||A||_F = \sqrt{\sum_{ij} |a_{ij}|^2} = \sqrt{\mathrm{tr}\,[A^\dagger A]} = \sqrt{\sum_{i=1}^{\mathrm{rank}(A)} \sigma_i^2}. \tag{2}$$

Here σ_i are the singular values of A. Put

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^\dagger. \tag{3}$$

The claim is the following.

With respect to both the operator norm and the Frobenius norm, the best approximation of A by a rank-k matrix is given by A_k. In other words, for any rank-k matrix B_k,

$$||A - B_k||_F \geq ||A - A_k||_F, \quad ||A - B_k||_2 \geq ||A - A_k||_2. \tag{4}$$

The following is a note on the proof in [1]. By Weyl's inequality, for arbitrary i, j,

$$\sigma_{i+j-1}(X + Y) \leq \sigma_i(X) + \sigma_j(Y). \tag{5}$$

Let the rank of B be k. Note that $\sigma_{i>k}(B) = 0$. Taking $j = k + 1$, $X = A - B$, and $Y = B$, we get

$$\sigma_{i+k}(A) \leq \sigma_i(A - B) + \sigma_{k+1}(B) = \sigma_i(A - B). \tag{6}$$

Thus

$$||A - B||_2 = \sigma_1(A - B) \geq \sigma_{k+1}(A) = \sigma_1(A - A_k), \tag{7}$$

and

$$||A - B||_F^2 = \sum_i \sigma_i^2(A - B) \geq \sum_i \sigma_{i+k}^2(A) = ||A - A_k||_F^2. \tag{8}$$

As a corollary, we obtain the following statement.

Let $|\psi\rangle$ be a normalized state with Schmidt decomposition $|\psi\rangle = \sum_{i\geq 1} \lambda_i |L_i\rangle |R_i\rangle$. Let $|\phi_k\rangle$ be a normalized state of Schmidt rank k. Then

$$|\langle\psi|\phi_k\rangle| \leq \sum_{i\leq k} \lambda_i^2. \tag{9}$$

For the proof, let the rank-k approximation be

$$|\psi_k\rangle = \sum_{i\leq k} \lambda_i |L_i\rangle |R_i\rangle . \tag{10}$$

By (8),

$$|| \, |\psi\rangle - |\phi_k\rangle \, ||^2 \geq || \, |\psi\rangle - |\psi_k\rangle \, ||^2. \tag{11}$$

Therefore

$$2 - 2\text{Re} \langle\psi|\phi_k\rangle \geq 1 + \langle\psi_k|\psi_k\rangle - 2\text{Re} \langle\psi|\psi_k\rangle . \tag{12}$$

It follows that

$$\text{Re} \langle\psi|\phi_k\rangle \leq \text{Re} \langle\psi|\psi_k\rangle + 1 - \langle\psi_k|\psi_k\rangle /2 \leq \sum_{i\leq k} \lambda_i^2. \tag{13}$$

Since the phase of $|\phi_k\rangle$ is arbitrary, this proves the claim.

References

[1] Physics Stack Exchange, *Proof of Eckart-Young-Mirsky theorem*, url.