Memo: On Stable Framed Cobordism

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1 Memo

For an n-dimensional compact manifold M_n, a framing ψ means giving a global frame of TM:

$$\psi : TM \cong M \times \mathbb{R}^n. \tag{1}$$

Notice that a framing need not exist, and need not be unique. For instance, TS^2 does not admit a global frame. If two framings (M_n, ψ) and (M'_n, ψ') are given, then

$$\psi' \circ \psi^{-1} : M \times \mathbb{R}^n \to TM \to M \times \mathbb{R}^n \tag{2}$$

determines a map

$$g = \psi' \circ \psi^{-1} : M \to O(n). \tag{3}$$

Thus framings differ by the homotopy set

$$[M, O(n)]. \tag{4}$$

We say that (M_n, ψ) and (M'_n, ψ') are framed cobordant,

$$(M_n, \psi) \sim (M'_n, \psi'), \tag{5}$$

if there exists a manifold W_{n+1} with $\partial W_{n+1} = M_n \sqcup (-M'_n)$, and if one can choose a framing of TW whose restriction to the boundary agrees with (M_n, ψ) and (M'_n, ψ').

In stable framed cobordism, one asks whether two manifolds are framed cobordant after allowing direct sums with trivial bundles. Namely, saying that M_n has a stable framing means that, for some k,

$$TM \oplus \underline{\mathbb{R}}^k \tag{6}$$

admits a framing. For example, since $TS^2 \oplus \underline{\mathbb{R}} \cong TS^2 \oplus NS^2 \cong \underline{\mathbb{R}}^3$, the sphere S^2 has a stable framing. If (M, ψ) and (M', ψ') are cobordant in this stable sense, we call them stably framed cobordant. The equivalence classes are denoted by Ω^{fr}_n and form the framed cobordism group.

As a fact, the isomorphism classes of spin structures on S^1 are isomorphic to Ω^{fr}_1. Hence spin structures on S^1 can be described by stable framings. Let t be the coordinate on S^1. The R sector is given by

$$TS^1 \to \mathbb{R}\, \partial_t, \tag{7}$$

whereas the NS sector corresponds to choosing

$$TS^1 \oplus \underline{\mathbb{R}} \to \mathbb{R}\, \partial_t \oplus \mathbb{R}\, \boldsymbol{n} \cong TD^2|_{\partial D^2}. \tag{8}$$

Here $\boldsymbol{n}$ is the unit normal vector to S^1 at the point $t \in S^1$.