Ground States of the Kitaev Chain

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June 1, 2026

Abstract

I often need to derive these formulas, so I collect them here.

1 Kitaev chain

Let $a_j, a_j^\dagger$ be the annihilation and creation operators of a complex fermion defined on site j. We take the Hamiltonian to be [1]

$$H = \sum_j \left[-t(a_{j+1}^\dagger a_j + h.c.) - \mu(a_j^\dagger a_j - \frac{1}{2}) + \Delta(a_{j+1}^\dagger a_j^\dagger + h.c.) \right]. \tag{1}$$

Here t, μ are real and Δ is complex. [1] When we consider a closed chain with L sites, we may impose periodic boundary conditions (PBC) or antiperiodic boundary conditions (APBC). For PBC the boundary term is

$$-t(a_1^\dagger a_L + h.c.) + \Delta(a_1^\dagger a_L^\dagger + h.c.). \tag{2}$$

For APBC, on the other hand, it is the same term multiplied by (-1),

$$-[-t(a_1^\dagger a_L + h.c.) + \Delta(a_1^\dagger a_L^\dagger + h.c.)]. \tag{3}$$

Write $\Delta = |\Delta| e^{i\theta}, |\Delta| \geq 0$. After the redefinition

$$\tilde{a}_j^\dagger = e^{i\theta/2} a_j^\dagger, \tag{4}$$

the Hamiltonian becomes

$$H = \sum_j \left[-t(\tilde{a}_{j+1}^\dagger \tilde{a}_j + h.c.) - \mu(\tilde{a}_j^\dagger \tilde{a}_j - \frac{1}{2}) + |\Delta|(\tilde{a}_{j+1}^\dagger \tilde{a}_j^\dagger + h.c.) \right]. \tag{5}$$

Introduce Majorana fermions by

$$c_{2j-1} = \tilde{a}_j + \tilde{a}_j^\dagger = e^{-i\theta/2} a_j + e^{i\theta/2} a_j^\dagger, \tag{6}$$

$$c_{2j} = -i(\tilde{a}_j - \tilde{a}_j^\dagger) = -i(e^{-i\theta/2} a_j - e^{i\theta/2} a_j^\dagger), \tag{7}$$

or, equivalently,

$$\tilde{a}_j = \frac{c_{2j-1} + ic_{2j}}{2}, \quad \tilde{a}_j^\dagger = \frac{c_{2j-1} - ic_{2j}}{2}. \tag{8}$$

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[1] Compared with [1], the definitions of Δ and Δ^* are interchanged.

Now note that

$$\tilde{a}^\dagger_{j+1}\tilde{a}_j + h.c. = \frac{c_{2j+1} - ic_{2j+2}}{2}\frac{c_{2j-1} + ic_{2j}}{2} + h.c. = -\frac{i}{2}c_{2j}c_{2j+1} + \frac{i}{2}c_{2j-1}c_{2j+2}, \tag{9}$$

$$\tilde{a}^\dagger_{j+1}\tilde{a}^\dagger_j + h.c. = \frac{c_{2j+1} - ic_{2j+2}}{2}\frac{c_{2j-1} - ic_{2j}}{2} + h.c. = \frac{i}{2}c_{2j}c_{2j+1} + \frac{i}{2}c_{2j-1}c_{2j+2}, \tag{10}$$

$$\tilde{a}^\dagger_j\tilde{a}_j - \frac{1}{2} = \frac{c_{2j-1} - ic_{2j}}{2}\frac{c_{2j-1} + ic_{2j}}{2} - \frac{1}{2} = \frac{i}{2}c_{2j-1}c_{2j}. \tag{11}$$

Therefore the Hamiltonian is

$$\begin{aligned} H &= \sum_j \left[-t(-\frac{i}{2}c_{2j}c_{2j+1} + \frac{i}{2}c_{2j-1}c_{2j+2}) - \mu(\frac{i}{2}c_{2j-1}c_{2j}) + |\Delta|(\frac{i}{2}c_{2j}c_{2j+1} + \frac{i}{2}c_{2j-1}c_{2j+2})\right] && (12)\\ &= \frac{i}{2}\sum_j \left[-\mu c_{2j-1}c_{2j} + (t + |\Delta|)c_{2j}c_{2j+1} + (-t + |\Delta|)c_{2j-1}c_{2j+2}\right]. && (13)\end{aligned}$$

2 Ground states at zero correlation length

Below, for the Kitaev-chain Hamiltonian at

- $\mu = -1, t = |\Delta| = 0$,
- $\mu = 1, t = |\Delta| = 0$,
- $\mu = 0, t = |\Delta| = \frac{1}{2}$,
- $\mu = 0, t = -|\Delta| = -\frac{1}{2}$,

we determine, for both PBC and APBC, the expression of the ground state in terms of the vacuum of the complex fermions a_j,

$$a_j |0\rangle = 0, \quad j = 1, \dots, L, \tag{14}$$

and the creation operators $a^\dagger_j$.

2.1 $\mu = -1, t = |\Delta| = 0$

The Hamiltonian is

$$H^{(1)} = \frac{i}{2}\sum_j c_{2j-1}c_{2j} = \sum_j (a^\dagger_j a_j - \frac{1}{2}). \tag{15}$$

Independently of the boundary condition, the ground state is

$$|\Psi^{(1)}_{\mathrm{PBC/APBC}}\rangle = |0\rangle\,. \tag{16}$$

2.2 $\mu = 1, t = |\Delta| = 0$

The Hamiltonian is

$$H^{(2)} = -\frac{i}{2}\sum_j c_{2j-1}c_{2j} = -\sum_j (a_j^\dagger a_j - \frac{1}{2}). \tag{17}$$

Independently of the boundary condition, the ground state is

$$|\Psi^{(2)}_{\mathrm{PBC/APBC}}\rangle = a_1^\dagger \cdots a_L^\dagger |0\rangle . \tag{18}$$

2.3 $\mu = 0, t = |\Delta| = \frac{1}{2}$

The Hamiltonian is

$$\begin{aligned} H^{(3)} &= \frac{i}{2}\sum_{j=1}^{L-1} c_{2j}c_{2j+1} + \eta\frac{i}{2}c_{2L}c_1 && (19)\\ &= \frac{1}{2}\sum_{j=1}^{L-1}\Big[-(a_{j+1}^\dagger a_j + h.c.) + e^{i\theta}(a_{j+1}^\dagger a_j^\dagger + h.c.)\Big] + \eta\frac{1}{2}\Big[-(a_1^\dagger a_L + h.c.) + e^{i\theta}(a_1^\dagger a_L^\dagger + h.c.)\Big]. && (20) \end{aligned}$$

Here $\eta = \pm 1$ specifies the boundary condition. Introduce a complex fermion on each bond by

$$\begin{aligned} \tilde{b}_j &= \frac{c_{2j} + ic_{2j+1}}{2} = \frac{i}{2}(-\tilde{a}_j + \tilde{a}_j^\dagger + \tilde{a}_{j+1} + \tilde{a}_{j+1}^\dagger), && (21)\\ \tilde{b}_j^\dagger &= \frac{c_{2j} - ic_{2j+1}}{2} = -\frac{i}{2}(\tilde{a}_j - \tilde{a}_j^\dagger + \tilde{a}_{j+1}^\dagger + \tilde{a}_{j+1}), && (22)\\ j &= 1, \dots, L. && (23) \end{aligned}$$

Using

$$c_{2j} = \tilde{b}_j + \tilde{b}_j^\dagger, \quad c_{2j+1} = -i(\tilde{b}_j - \tilde{b}_j^\dagger), \tag{24}$$

the Hamiltonian becomes

$$H^{(3)} = \sum_{j=1}^{L-1}(\tilde{b}_j^\dagger \tilde{b}_j - \frac{1}{2}) + \eta(\tilde{b}_L^\dagger \tilde{b}_L - \frac{1}{2}). \tag{25}$$

Independently of the boundary condition, the ground state satisfies

$$\tilde{b}_j |\Psi^{(3)}\rangle = 0, \quad j = 1, \dots, L-1. \tag{26}$$

There are two states satisfying this condition, given by

$$|\pm\rangle = (1 \pm \tilde{a}_1^\dagger)\cdots(1 \pm \tilde{a}_L^\dagger)|0\rangle = \sum_{n=1}^{L}\sum_{j_1 < \cdots < j_n}(\pm 1)^n \tilde{a}_{j_1}^\dagger \cdots \tilde{a}_{j_n}^\dagger |0\rangle . \tag{27}$$

[2] Indeed,

$$\begin{aligned} (-\tilde{a}_j + \tilde{a}_j^\dagger)|\pm\rangle &= (1 \mp \tilde{a}_1^\dagger)\cdots(1 \mp \tilde{a}_{j-1}^\dagger)(-\tilde{a}_j + \tilde{a}_j^\dagger)(1 \pm \tilde{a}_j^\dagger)\cdots(1 \pm \tilde{a}_L^\dagger)|0\rangle && (28)\\ &= (1 \mp \tilde{a}_1^\dagger)\cdots(1 \mp \tilde{a}_{j-1}^\dagger)(-\tilde{a}_j \mp \tilde{a}_j\tilde{a}_j^\dagger + \tilde{a}_j^\dagger)(1 \pm \tilde{a}_{j+1}^\dagger)\cdots(1 \pm \tilde{a}_L^\dagger)|0\rangle && (29)\\ &= (1 \mp \tilde{a}_1^\dagger)\cdots(1 \mp \tilde{a}_{j-1}^\dagger)(\mp 1 + \tilde{a}_j^\dagger)(1 \pm \tilde{a}_{j+1}^\dagger)\cdots(1 \pm \tilde{a}_L^\dagger)|0\rangle && (30)\\ &= \mp(1 \mp \tilde{a}_1^\dagger)\cdots(1 \mp \tilde{a}_{j-1}^\dagger)(1 \mp \tilde{a}_j^\dagger)(1 \pm \tilde{a}_{j+1}^\dagger)\cdots(1 \pm \tilde{a}_L^\dagger)|0\rangle , && (31) \end{aligned}$$

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[2] Under the Jordan-Wigner transformation this becomes the Ising model, and these states are the all-up and all-down states.

$$\begin{aligned}(\tilde{a}_j+\tilde{a}_j^\dagger)\left|\pm\right\rangle &= (1\mp\tilde{a}_1^\dagger)\cdots(1\mp\tilde{a}_{j-1}^\dagger)(\tilde{a}_j+\tilde{a}_j^\dagger)(1\pm\tilde{a}_j^\dagger)\cdots(1\pm\tilde{a}_L^\dagger)\left|0\right\rangle && (32)\\ &= (1\mp\tilde{a}_1^\dagger)\cdots(1\mp\tilde{a}_{j-1}^\dagger)(\tilde{a}_j\pm\tilde{a}_j\tilde{a}_j^\dagger+\tilde{a}_j^\dagger)(1\pm\tilde{a}_{j+1}^\dagger)\cdots(1\pm\tilde{a}_L^\dagger)\left|0\right\rangle && (33)\\ &= (1\mp\tilde{a}_1^\dagger)\cdots(1\mp\tilde{a}_{j-1}^\dagger)(\pm1+\tilde{a}_j^\dagger)(1\pm\tilde{a}_{j+1}^\dagger)\cdots(1\pm\tilde{a}_L^\dagger)\left|0\right\rangle && (34)\\ &= \pm(1\mp\tilde{a}_1^\dagger)\cdots(1\mp\tilde{a}_{j-1}^\dagger)(1\pm\tilde{a}_j^\dagger)(1\pm\tilde{a}_{j+1}^\dagger)\cdots(1\pm\tilde{a}_L^\dagger)\left|0\right\rangle. && (35)\end{aligned}$$

It follows that

$$(-\tilde{a}_j^\dagger+\tilde{a}_j+\tilde{a}_{j+1}+\tilde{a}_{j+1}^\dagger)\left|\pm\right\rangle = 0, \quad j=1,\dots,L-1. \tag{36}$$

The boundary term closes within the space spanned by $\left|+\right\rangle, \left|-\right\rangle$. Since

$$\begin{aligned}\tilde{b}_L\left|\pm\right\rangle &= \frac{i}{2}\{(-\tilde{a}_L+\tilde{a}_L^\dagger)+(\tilde{a}_1+\tilde{a}_1^\dagger)\}\left|\pm\right\rangle = \frac{i}{2}\{\mp\left|\mp\right\rangle\pm\left|\pm\right\rangle\} = \pm\frac{i}{2}(\left|\pm\right\rangle-\left|\mp\right\rangle), && (37)\\ \tilde{b}_L^\dagger\left|\pm\right\rangle &= -\frac{i}{2}\{(\tilde{a}_L-\tilde{a}_L^\dagger)+(\tilde{a}_1+\tilde{a}_1^\dagger)\}\left|\pm\right\rangle = -\frac{i}{2}\{\pm\left|\mp\right\rangle\pm\left|\pm\right\rangle\} = \mp\frac{i}{2}(\left|\mp\right\rangle+\left|\pm\right\rangle), && (38)\end{aligned}$$

we have

$$\tilde{b}_L^\dagger\tilde{b}_L\left|\pm\right\rangle = \pm\frac{i}{2}\tilde{b}_L^\dagger(\left|\pm\right\rangle-\left|\mp\right\rangle) = \pm\frac{i}{2}\{\mp\frac{i}{2}(\left|\mp\right\rangle+\left|\pm\right\rangle)\mp\frac{i}{2}(\left|\pm\right\rangle+\left|\mp\right\rangle)\} = \frac{1}{2}(\left|\pm\right\rangle+\left|\mp\right\rangle). \tag{39}$$

In other words,

$$\eta(\tilde{b}_L^\dagger\tilde{b}_L-\frac{1}{2})\left|\pm\right\rangle = \eta\frac{1}{2}\left|\mp\right\rangle. \tag{40}$$

Thus, up to normalization, the ground states of $H^{(3)}$ are respectively

$$\begin{aligned}|\Psi^{(3)}_{\mathrm{PBC}}\rangle &\sim \frac{\left|+\right\rangle-\left|-\right\rangle}{2} = \sum_{n\in\mathrm{odd}}\sum_{j_1<\cdots<j_n}\tilde{a}_{j_1}^\dagger\cdots\tilde{a}_{j_n}^\dagger\left|0\right\rangle = \sum_{n\in\mathrm{odd}}\sum_{j_1<\cdots<j_n}e^{in\theta/2}a_{j_1}^\dagger\cdots a_{j_n}^\dagger\left|0\right\rangle, && (41)\\ |\Psi^{(3)}_{\mathrm{APBC}}\rangle &\sim \frac{\left|+\right\rangle+\left|-\right\rangle}{2} = \sum_{n\in\mathrm{even}}\sum_{j_1<\cdots<j_n}\tilde{a}_{j_1}^\dagger\cdots\tilde{a}_{j_n}^\dagger\left|0\right\rangle = \sum_{n\in\mathrm{even}}\sum_{j_1<\cdots<j_n}e^{in\theta/2}a_{j_1}^\dagger\cdots a_{j_n}^\dagger\left|0\right\rangle. && (42)\end{aligned}$$

2.4 $\mu=0, t=-|\Delta|=-\frac{1}{2}$

The Hamiltonian is

$$\begin{aligned}H^{(4)} &= \frac{i}{2}\sum_{j=1}^{L-1}c_{2j-1}c_{2j+2}+\eta\frac{i}{2}c_{2L-1}c_2 && (43)\\ &= \frac{1}{2}\sum_{j=1}^{L-1}\left[(a_{j+1}^\dagger a_j+h.c.)+e^{i\theta}(a_{j+1}^\dagger a_j^\dagger+h.c.)\right]+\eta\frac{1}{2}[(a_1^\dagger a_L+h.c.)+e^{i\theta}(a_1^\dagger a_L^\dagger+h.c.)]. && (44)\end{aligned}$$

$H^{(4)}$ is obtained from $H^{(3)}$ by a fermion-parity transformation on the odd sites together with a phase transformation by i:

$$U=\prod_{j=1}^{L}i^{a_j^\dagger a_j}(-1)^{ja_j^\dagger a_j}. \tag{45}$$

Then

$$UH^{(3)}U^{-1} = \frac{1}{2}\sum_{j=1}^{L-1}\left[(a_{j+1}^\dagger a_j+h.c.)+e^{i\theta}(a_{j+1}^\dagger a_j^\dagger+h.c.)\right]+(-1)^L\eta\frac{1}{2}\left[(a_1^\dagger a_L+h.c.)+e^{i\theta}(a_1^\dagger a_L^\dagger+h.c.)\right]. \tag{46}$$

Therefore the ground states of $H^{(4)}$ are given by

$$|\Psi^{(4)}_{\mathrm{PBC/APBC}}\rangle = \begin{cases} U\,|\Psi^{(3)}_{\mathrm{PBC/APBC}}\rangle & (L \in \text{even}) \\ U\,|\Psi^{(3)}_{\mathrm{APBC/PBC}}\rangle & (L \in \text{odd}) \end{cases} \tag{47}$$

$$\tag{48}$$

and explicitly

$$|\Psi^{(4)}_{\mathrm{PBC}}\rangle = \begin{cases} \sum_{n\in\mathrm{odd}} \sum_{j_1<\cdots<j_n} (-1)^{j_1+\cdots+j_n} i^n e^{in\theta/2} \tilde{a}^\dagger_{j_1} \cdots \tilde{a}^\dagger_{j_n} |0\rangle & (L \in \text{even}) \\ \sum_{n\in\mathrm{even}} \sum_{j_1<\cdots<j_n} (-1)^{j_1+\cdots+j_n} i^n e^{in\theta/2} \tilde{a}^\dagger_{j_1} \cdots \tilde{a}^\dagger_{j_n} |0\rangle & (L \in \text{odd}) \end{cases} \tag{49}$$

$$|\Psi^{(4)}_{\mathrm{APBC}}\rangle = \begin{cases} \sum_{n\in\mathrm{even}} \sum_{j_1<\cdots<j_n} (-1)^{j_1+\cdots+j_n} i^n e^{in\theta/2} \tilde{a}^\dagger_{j_1} \cdots \tilde{a}^\dagger_{j_n} |0\rangle & (L \in \text{even}) \\ \sum_{n\in\mathrm{odd}} \sum_{j_1<\cdots<j_n} (-1)^{j_1+\cdots+j_n} i^n e^{in\theta/2} \tilde{a}^\dagger_{j_1} \cdots \tilde{a}^\dagger_{j_n} |0\rangle & (L \in \text{odd}) \end{cases} \tag{50}$$

[3]

2.5 Fermion parity and momentum

For the ground states computed above, the fermion parity is

$$(-1)^F = \prod_{j=1}^{L} (-1)^{a^\dagger_j a_j}. \tag{51}$$

Thus

$$(-1)^F\,|\Psi^{(1)}_{\mathrm{PBC/APBC}}\rangle = |\Psi^{(1)}_{\mathrm{PBC/APBC}}\rangle\,, \tag{52}$$

$$(-1)^F\,|\Psi^{(2)}_{\mathrm{PBC/APBC}}\rangle = (-1)^L\,|\Psi^{(2)}_{\mathrm{PBC/APBC}}\rangle\,, \tag{53}$$

$$(-1)^F\,|\Psi^{(3)}_{\mathrm{PBC/APBC}}\rangle = \mp\,|\Psi^{(3)}_{\mathrm{PBC/APBC}}\rangle\,, \tag{54}$$

$$(-1)^F\,|\Psi^{(4)}_{\mathrm{PBC/APBC}}\rangle = \mp(-1)^L\,|\Psi^{(4)}_{\mathrm{PBC/APBC}}\rangle\,. \tag{55}$$

On the other hand, define the translation operators for PBC and APBC by

$$T_{\mathrm{PBC}} a^\dagger_j T^{-1}_{\mathrm{PBC}} = a^\dagger_{j+1}, \quad T_{\mathrm{PBC}} a^\dagger_L T^{-1}_{\mathrm{PBC}} = a^\dagger_1, \quad T_{\mathrm{PBC}}\,|0\rangle = |0\rangle\,, \tag{56}$$

$$T_{\mathrm{APBC}} a^\dagger_j T^{-1}_{\mathrm{APBC}} = a^\dagger_{j+1}, \quad T_{\mathrm{APBC}} a^\dagger_L T^{-1}_{\mathrm{APBC}} = -a^\dagger_1, \quad T_{\mathrm{APBC}}\,|0\rangle = |0\rangle\,. \tag{57}$$

[4] Then

$$T_{\mathrm{PBC/APBC}}\,|\Psi^{(1)}_{\mathrm{PBC/APBC}}\rangle = |\Psi^{(1)}_{\mathrm{PBC/APBC}}\rangle\,, \tag{58}$$

$$T_{\mathrm{PBC/APBC}}\,|\Psi^{(2)}_{\mathrm{PBC/APBC}}\rangle = \pm(-1)^{L-1}\,|\Psi^{(2)}_{\mathrm{PBC/APBC}}\rangle\,. \tag{59}$$

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[3] The factor $(-1)^{j_1+\cdots j_n}$ corresponds to the ground state of the antiferromagnetic Ising chain.

[4] The states $|\Psi^{(1)}\rangle$, $|\Psi^{(2)}\rangle$ are eigenstates of both T_{PBC} and T_{APBC} independently of the boundary condition, but this is a special feature of zero correlation length. For a general Hamiltonian, the Hamiltonian with PBC/APBC is invariant only under $T_{\mathrm{PBC}}/T_{\mathrm{APBC}}$.

For the $\mathbb{Z}_2$-nontrivial state,

$$T_{\mathrm{PBC}}\,|\Psi^{(3)}_{\mathrm{PBC}}\rangle = T_{\mathrm{PBC}}\left[\sum_{n\in\mathrm{odd}}\sum_{j_1<\cdots<j_n<L} e^{in\theta/2}a^\dagger_{j_1}\cdots a^\dagger_{j_n}\,|0\rangle + \sum_{n\in\mathrm{odd}}\sum_{j_1<\cdots<j_n=L} e^{in\theta/2}a^\dagger_{j_1}\cdots a^\dagger_{L}\,|0\rangle\right] \tag{60}$$

$$= \sum_{n\in\mathrm{odd}}\sum_{j_1<\cdots<j_n<L} e^{in\theta/2}a^\dagger_{j_1+1}\cdots a^\dagger_{j_n+1}\,|0\rangle + \sum_{n\in\mathrm{odd}}\sum_{j_1<\cdots<j_n=L} e^{in\theta/2}a^\dagger_{j_1+1}\cdots a^\dagger_{1}\,|0\rangle \tag{61}$$

$$= \sum_{n\in\mathrm{odd}}\sum_{j_1<\cdots<j_n<L} e^{in\theta/2}a^\dagger_{j_1+1}\cdots a^\dagger_{j_n+1}\,|0\rangle + \sum_{n\in\mathrm{odd}}\sum_{j_1<\cdots<j_n=L} e^{in\theta/2}a^\dagger_{1}a^\dagger_{j_1+1}\cdots a^\dagger_{j_{n-1}}\,|0\rangle \tag{62}$$

$$= |\Psi^{(3)}_{\mathrm{PBC}}\rangle\,, \tag{63}$$

$$T_{\mathrm{APBC}}\,|\Psi^{(3)}_{\mathrm{APBC}}\rangle = T_{\mathrm{APBC}}\left[\sum_{n\in\mathrm{even}}\sum_{j_1<\cdots<j_n<L} e^{in\theta/2}a^\dagger_{j_1}\cdots a^\dagger_{j_n}\,|0\rangle + \sum_{n\in\mathrm{even}}\sum_{j_1<\cdots<j_n=L} e^{in\theta/2}a^\dagger_{j_1}\cdots a^\dagger_{L}\,|0\rangle\right] \tag{64}$$

$$= \sum_{n\in\mathrm{even}}\sum_{j_1<\cdots<j_n<L} e^{in\theta/2}a^\dagger_{j_1+1}\cdots a^\dagger_{j_n+1}\,|0\rangle + \sum_{n\in\mathrm{even}}\sum_{j_1<\cdots<j_n=L} e^{in\theta/2}a^\dagger_{j_1+1}\cdots(-a^\dagger_{1})\,|0\rangle \tag{65}$$

$$= \sum_{n\in\mathrm{even}}\sum_{j_1<\cdots<j_n<L} e^{in\theta/2}a^\dagger_{j_1+1}\cdots a^\dagger_{j_n+1}\,|0\rangle + \sum_{n\in\mathrm{even}}\sum_{j_1<\cdots<j_n=L} e^{in\theta/2}a^\dagger_{1}a^\dagger_{j_1+1}\cdots a^\dagger_{j_{n-1}}\,|0\rangle \tag{66}$$

$$= |\Psi^{(3)}_{\mathrm{APBC}}\rangle\,. \tag{67}$$

Therefore

$$T_{\mathrm{PBC/APBC}}\,|\Psi^{(3)}_{\mathrm{PBC/APBC}}\rangle = |\Psi^{(3)}_{\mathrm{PBC/APBC}}\rangle\,. \tag{68}$$

Let us compute the momentum of $|\Psi^{(4)}_{\mathrm{PBC/APBC}}\rangle$ from the relation between translation and U.

$$T_{\mathrm{PBC/APBC}}U = \Big(\prod_{j=1}^{L} i^{\hat{n}_{j+1}}(-1)^{j\hat{n}_{j+1}}\Big)T_{\mathrm{PBC/APBC}}. \tag{69}$$

Here

$$\prod_{j=1}^{L}(-1)^{j\hat{n}_{j+1}} = \begin{cases} (-1)^{\hat{n}_2}(-1)^{\hat{n}_4}\cdots(-1)^{\hat{n}_L} & (L\in\mathrm{even}) \\ (-1)^{\hat{n}_2}(-1)^{\hat{n}_4}\cdots(-1)^{\hat{n}_{L-1}}(-1)^{\hat{n}_1} & (L\in\mathrm{odd}) \end{cases} \tag{70}$$

$$= \begin{cases} (\prod_{j=1}^{L}(-1)^{j\hat{n}_j})(-1)^F & (L\in\mathrm{even}) \\ (\prod_{j=1}^{L}(-1)^{j\hat{n}_j})(-1)^F(-1)^{\hat{n}_1} & (L\in\mathrm{odd}) \end{cases} \tag{71}$$

and hence

$$T_{\mathrm{PBC/APBC}}U = \begin{cases} U(-1)^F T_{\mathrm{PBC/APBC}} & (L\in\mathrm{even}) \\ U(-1)^F(-1)^{\hat{n}_1}T_{\mathrm{PBC/APBC}} & (L\in\mathrm{odd}) \end{cases} \tag{72}$$

Furthermore, using

$$(-1)^{\hat{n}_1}T_{\mathrm{PBC/APBC}} = T_{\mathrm{APBC/PBC}}, \tag{73}$$

we finally get

$$T_{\mathrm{PBC/APBC}}U = U(-1)^F\begin{cases} T_{\mathrm{PBC/APBC}} & (L\in\mathrm{even}) \\ T_{\mathrm{APBC/PBC}} & (L\in\mathrm{odd}) \end{cases}\,. \tag{74}$$

Thus

$$T_{\text{PBC/APBC}}\,|\Psi^{(4)}_{\text{PBC/APBC}}\rangle = \begin{cases} T_{\text{PBC/APBC}}U\,|\Psi^{(3)}_{\text{PBC/APBC}}\rangle & (L\in\text{even}) \\ T_{\text{PBC/APBC}}U\,|\Psi^{(3)}_{\text{APBC/PBC}}\rangle & (L\in\text{odd}) \end{cases} \tag{75}$$

$$= \begin{cases} U(-1)^F T_{\text{PBC/APBC}}\,|\Psi^{(3)}_{\text{PBC/APBC}}\rangle & (L\in\text{even}) \\ U(-1)^F T_{\text{APBC/PBC}}\,|\Psi^{(3)}_{\text{APBC/PBC}}\rangle & (L\in\text{odd}) \end{cases} \tag{76}$$

$$= \begin{cases} U(-1)^F\,|\Psi^{(3)}_{\text{PBC/APBC}}\rangle & (L\in\text{even}) \\ U(-1)^F\,|\Psi^{(3)}_{\text{APBC/PBC}}\rangle & (L\in\text{odd}) \end{cases} \tag{77}$$

$$= \begin{cases} \mp U\,|\Psi^{(3)}_{\text{PBC/APBC}}\rangle & (L\in\text{even}) \\ \pm U\,|\Psi^{(3)}_{\text{APBC/PBC}}\rangle & (L\in\text{odd}) \end{cases} \tag{78}$$

$$= \pm(-1)^{L-1}\,|\Psi^{(4)}_{\text{PBC/APBC}}\rangle\,. \tag{79}$$

Collecting all results,

$$T_{\text{PBC/APBC}}\,|\Psi^{(1)}_{\text{PBC/APBC}}\rangle = |\Psi^{(1)}_{\text{PBC/APBC}}\rangle\,, \tag{80}$$

$$T_{\text{PBC/APBC}}\,|\Psi^{(2)}_{\text{PBC/APBC}}\rangle = \pm(-1)^{L-1}\,|\Psi^{(2)}_{\text{PBC/APBC}}\rangle\,, \tag{81}$$

$$T_{\text{PBC/APBC}}\,|\Psi^{(3)}_{\text{PBC/APBC}}\rangle = |\Psi^{(3)}_{\text{PBC/APBC}}\rangle\,, \tag{82}$$

$$T_{\text{PBC/APBC}}\,|\Psi^{(4)}_{\text{PBC/APBC}}\rangle = \pm(-1)^{L-1}\,|\Psi^{(4)}_{\text{PBC/APBC}}\rangle\,. \tag{83}$$

References

[1] A. Y. Kitaev, Physics-Uspekhi **44**, 131 (2001).