Irreducible Decomposition of a Superconducting Gap Function

Ken Shiozaki

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Consider the mean-field many-body Hamiltonian for a superconducting gap function,

$$\hat{\Delta}(\boldsymbol{k}) = \frac{1}{2} \sum_{\boldsymbol{k},i,j} \psi_i^\dagger(\boldsymbol{k}) \Delta_{ij}(\boldsymbol{k}) \psi_j^\dagger(-\boldsymbol{k}) + h.c. \tag{1}$$

Assume that the normal state has unitary symmetries of a group G:

$$\hat{g} \psi_i^\dagger(\boldsymbol{k}) \hat{g}^{-1} = \psi_j^\dagger(p_g \boldsymbol{k}) [u_g]_{ji}, \quad u_g u_h = z_{g,h} u_{gh}, \quad \hat{g} i \hat{g}^{-1} = i, \quad g, h \in G. \tag{2}$$

Then

$$\hat{g} \hat{\Delta} \hat{g}^{-1} = \frac{1}{2} \sum_{\boldsymbol{k},i,j} \psi_i^\dagger(p_g \boldsymbol{k}) [u_g \Delta(\boldsymbol{k}) u_g^T]_{ij} \psi_j^\dagger(-p_g \boldsymbol{k}) + h.c. \tag{3}$$

$$= \frac{1}{2} \sum_{\boldsymbol{k},i,j} \psi_i^\dagger(\boldsymbol{k}) [u_g \Delta(p_g^{-1} \boldsymbol{k}) u_g^T]_{ij} \psi_j^\dagger(-\boldsymbol{k}) + h.c. \tag{4}$$

Thus we introduce the left action of G on the gap function $\Delta(\boldsymbol{k})$ by

$$(D_g \Delta)(\boldsymbol{k}) := u_g \Delta(p_g^{-1} \boldsymbol{k}) u_g^T. \tag{5}$$

It satisfies

$$(D_g(D_h \Delta))(\boldsymbol{k}) = u_g (D_h \Delta)(p_g^{-1} \boldsymbol{k}) u_g^T \tag{6}$$

$$= u_g u_h \Delta(p_h^{-1} p_g^{-1} \boldsymbol{k}) u_h^T u_g^T \tag{7}$$

$$= (z_{g,h})^2 u_{gh} \Delta(p_{gh}^{-1} \boldsymbol{k}) u_{gh}^T \tag{8}$$

$$= (z_{g,h})^2 (D_{gh} \Delta)(\boldsymbol{k}). \tag{9}$$

Therefore, if the factor system obeys $(z_{g,h})^2 \equiv 1$, then $D_g D_h = D_{gh}$.

The irreducible decomposition of a given superconducting gap function under G is expected to be obtained from the projection formula onto an irreducible representation α of G,

$$P_\alpha = \frac{\dim \alpha}{|G|} \sum_{g \in G} (\chi_g^\alpha)^* D_g. \tag{10}$$

Namely,

$$\Delta(\boldsymbol{k}) = \sum_{\alpha \in \mathrm{irreps}} \Delta^{(\alpha)}(\boldsymbol{k}), \tag{11}$$

$$\Delta^{(\alpha)}(\boldsymbol{k}) = \frac{\dim \alpha}{|G|} \sum_{g \in G} (\chi_g^\alpha)^* (D_g \Delta)(\boldsymbol{k}). \tag{12}$$

Here χ_g^α denotes the irreducible character.