On the Gauge Invariance of c_1 and w_2

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1 The First Chern Class

Consider a complex line bundle $L \to X$. For a good cover $\{U_i\}_i$, let v_i be a local trivialization on U_i. On U_{ij} the transition function is

$$e^{i\phi_{ij}} = v_i^\dagger v_j \in U(1). \tag{1}$$

Choose a lift

$$\mathbb{R}/2\pi\mathbb{Z} \ni \phi_{ij} \to \tilde{\phi}_{ij} \in \mathbb{R}. \tag{2}$$

The first Chern class is defined as the cohomology class

$$c_1 = \left[\frac{1}{2\pi}\delta\tilde{\phi}\right] \in H^2(X,\mathbb{Z}) \tag{3}$$

of the 2-cocycle

$$(\delta\tilde{\phi})_{ijk} = \tilde{\phi}_{jk} - \tilde{\phi}_{ik} + \tilde{\phi}_{ij} \in Z^2(X, 2\pi\mathbb{Z}). \tag{4}$$

Changing the lift by

$$\tilde{\phi}_{ij} \mapsto \tilde{\phi}_{ij} + 2\pi m_{ij}, \quad m_{ij} \in \mathbb{Z} \tag{5}$$

does not change the cohomology class c_1.

Let us check the invariance of c_1 under gauge transformations. Under

$$v_i \mapsto v_i e^{i\chi_i}, \tag{6}$$

the transition functions transform as

$$e^{i\phi_{ij}} \mapsto e^{i\phi'_{ij}} = e^{i(\phi_{ij} - \chi_i + \chi_j)}. \tag{7}$$

A lift of ϕ'_{ij},

$$\mathbb{R}/2\pi\mathbb{Z} \ni \phi'_{ij} \to \tilde{\phi}'_{ij} \in \mathbb{R}, \tag{8}$$

can be chosen independently of the lift of ϕ_{ij}. However, from

$$e^{i\phi'_{ij}} = e^{i\phi_{ij}} e^{-i\chi_i} e^{i\chi_j}, \tag{9}$$

if we choose lifts of the gauge transformations

$$\mathbb{R}/2\pi\mathbb{Z} \ni \chi_i \to \tilde{\chi}_i \in \mathbb{R}, \tag{10}$$

then

$$\tilde{\phi}'_{ij} = \tilde{\phi}_{ij} - \tilde{\chi}_i + \tilde{\chi}_j + 2\pi n_{ij}, \quad n_{ij} \in \mathbb{Z}. \tag{11}$$

Therefore

$$\delta\tilde{\phi}' = \delta\tilde{\phi} + 2\pi\delta n. \tag{12}$$

Since $\delta n \in B^2(X,\mathbb{Z})$, the cohomology class c_1 is gauge invariant.

2 The Second Stiefel–Whitney Class

The argument is analogous to the one for c_1. Consider a rank-r real bundle $E \to X$. For a good cover $\{U_i\}_i$, let v_i be a local trivialization on U_i. On U_{ij} the transition function is

$$g_{ij} = v_i^\dagger v_j \in O(r). \tag{13}$$

Choose a lift

$$O(r) \ni g_{ij} \to \tilde{g}_{ij} \in Pin_+(r). \tag{14}$$

The second Stiefel–Whitney class is defined as the cohomology class $w_2 = [\delta\tilde{g}] \in H^2(X, \mathbb{Z}_2)$ of the 2-cocycle

$$(\delta\tilde{g})_{ijk} = \tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki} \in \{\pm 1\} \in Z^2(X, \mathbb{Z}_2). \tag{15}$$

Changing the lift by

$$\tilde{g}_{ij} \mapsto \eta_{ij}\tilde{g}_{ij}, \quad \eta_{ij} \in \{\pm 1\} \tag{16}$$

does not change the cohomology class w_2.

Let us check the invariance of w_2 under gauge transformations. Under

$$v_i \mapsto v_i V_i, \quad V_i \in O(r), \tag{17}$$

the transition functions transform as

$$g_{ij} \mapsto g'_{ij} = V_i^\dagger g_{ij} V_j. \tag{18}$$

A lift of g'_{ij},

$$O(r) \ni g'_{ij} \to \tilde{g}'_{ij} \in Pin_+(r), \tag{19}$$

can be chosen independently of the lift of g_{ij}. However, from

$$g'_{ij} = V_i^\dagger g_{ij} V_j, \tag{20}$$

if we choose lifts of the gauge transformations

$$O(r) \ni V_i \to \tilde{V}_i \in Pin_+(r), \tag{21}$$

then

$$\tilde{g}'_{ij} = \tilde{V}_i^\dagger \tilde{g}_{ij} \tilde{V}_j \times \eta_{ij}, \quad \eta_{ij} \in \{\pm 1\}. \tag{22}$$

Therefore

$$\delta\tilde{g}' = \delta\tilde{g} \times \delta\eta. \tag{23}$$

Since $\delta\eta \in B^2(X, \mathbb{Z}_2)$, the cohomology class w_2 is gauge invariant.