A Gauged Expression for the Three-Dimensional Winding Number

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1 Gauging

Let M be a closed oriented 3-manifold, and consider a map

$$g : M \to U(n). \tag{1}$$

The winding number is

$$\frac{1}{24\pi^2}\int_M \mathrm{tr}[g^{-1}dg]^3 \in \mathbb{Z}. \tag{2}$$

We gauge this topological action and study its behavior after coupling it to a background field. The action

$$H = \frac{1}{12\pi}\mathrm{tr}[g^{-1}dg]^3 \tag{3}$$

has a global $U(n)_L \times U(n)_R/U(1)_{\mathrm{diag}}$ symmetry

$$g \mapsto LgR^\dagger, \quad L, R \in U(n). \tag{4}$$

1.1 Local Expression

First ignore $U(1)_{\mathrm{diag}}$ and regard the symmetry as $U(n)_L \times U(n)_R$. Under a gauge transformation

$$g \mapsto g' = LgR^{-1}, \quad L, R \in U(n), \tag{5}$$

the exterior derivative transforms as

$$dg \mapsto d(LgR^{-1}) = L(dg + L^{-1}dLg - gR^{-1}dR)R^{-1}. \tag{6}$$

Introduce a $U(n)_L \times U(n)_R$ gauge field $A = (A_L, A_R)$ and define the covariant derivative by

$$D_A g = dg + A_L g - gA_R. \tag{7}$$

For $D_A g$ to transform covariantly,

$$D_A g \mapsto D_{A'} g' = LD_A gR^{-1}, \tag{8}$$

we define the gauge transformation of A by

$$A_L \mapsto A'_L = L(A_L + d)L^{-1} = LA_L L^{-1} + LdL^{-1}, \tag{9}$$

$$A_R \mapsto A'_R = R(A_R + d)R^{-1} = RA_R R^{-1} + RdR^{-1}. \tag{10}$$

Notice that

$$g^{-1}D_A g \mapsto R(g^{-1}D_A g)R^{-1}, \tag{11}$$

$$gD_A g^{-1} \mapsto L(gD_A g^{-1})L^{-1}. \tag{12}$$

In this sense,

$$\mathrm{tr}[g'^{-1}D_{A'}g']^3 = \mathrm{tr}[g^{-1}D_A g]^3 \tag{13}$$

is gauge invariant.

Let

$$A = g^{-1}dg, \quad B = g^{-1}A_L g, \quad C = -A_R. \tag{14}$$

Then

$$\mathrm{tr}[g^{-1}D_A g]^3 = \mathrm{tr}[A + B + C]^3. \tag{15}$$

A direct calculation gives

$$\begin{aligned}
&\mathrm{tr}[g^{-1}D_A g]^3 &(16)\\
&= \mathrm{tr}[(g^{-1}dg)^3 + A_L^3 - A_R^3] &(17)\\
&\quad + 3\mathrm{tr}\Big[dgg^{-1}F_L + g^{-1}dgF_R - F_L g A_R g^{-1} + F_R g^{-1} A_L g\Big] &(18)\\
&\quad + 3d\mathrm{tr}\Big[(dgg^{-1})A_L + (g^{-1}dg)A_R + A_L g A_R g^{-1}\Big], &(19)
\end{aligned}$$

where

$$F_{L/R} = dA_{L/R} + A_{L/R}^2 \tag{20}$$

is the curvature. Under gauge transformations,

$$F_L \mapsto L F_L L^{-1}, \quad F_R \mapsto R F_R R^{-1}. \tag{21}$$

Rearranging the result,

$$\begin{aligned}
\frac{1}{12\pi}\mathrm{tr}[g^{-1}D_A g]^3 &= \frac{1}{12\pi}\mathrm{tr}[g^{-1}dg]^3 + \frac{1}{4\pi}\mathrm{tr}\left[(g^{-1}D_{A_L}g)F_R - \frac{1}{3}A_R^3\right] &(22)\\
&\quad - \frac{1}{4\pi}\mathrm{tr}\left[(gD_{A_R}g^{-1})F_L - \frac{1}{3}A_L^3\right] &(23)\\
&\quad + \frac{1}{4\pi}d\mathrm{tr}\Big[(dgg^{-1})A_L + (g^{-1}dg)A_R + A_L g A_R g^{-1}\Big], &(24)
\end{aligned}$$

where $D_{A_{L/R}} = d + A_{L/R}$.

Extracting the gauge-invariant term

$$\mathrm{tr}[(g^{-1}D_A g)F_R - (gD_A g^{-1})F_L], \tag{25}$$

one obtains the expression

$$\boxed{\begin{aligned}
\frac{1}{12\pi}\mathrm{tr}[g^{-1}D_A g]^3 &= \frac{1}{12\pi}\mathrm{tr}[g^{-1}dg]^3 + \frac{1}{4\pi}\mathrm{tr}[(g^{-1}D_A g)F_R - (gD_A g^{-1})F_L] &(26)\\
&\quad + \frac{1}{4\pi}\mathrm{tr}\left[A_R F_R - \frac{1}{3}A_R^3\right] - \frac{1}{4\pi}\mathrm{tr}\left[A_L F_L - \frac{1}{3}A_L^3\right] &(27)\\
&\quad + \frac{1}{4\pi}d\mathrm{tr}\Big[(dgg^{-1})A_L - (dg^{-1}g)A_R + A_L g A_R g^{-1}\Big]. &(28)
\end{aligned}}$$

The total derivative term may still contribute under patch transformations.

1.1.1 Expression Using Chern–Simons Terms

Recall that the Chern–Simons term

$$CS(A) = \frac{1}{4\pi}\mathrm{tr}\left[AF - \frac{1}{3}A^3\right] \tag{29}$$

transforms under

$$A \mapsto A' = v^{-1}(d + A)v \tag{30}$$

as

$$CS(A') = CS(A) - \frac{1}{4\pi}d\mathrm{tr}[dvv^{-1}A] - \frac{1}{12\pi}\mathrm{tr}[dvv^{-1}]^3. \tag{31}$$

Using this, the previous formula can also be written as

$$\frac{1}{12\pi}\mathrm{tr}[g^{-1}D_A g]^3 = CS\big(g(A_R + d)g^{-1}\big) - CS\big(g^{-1}(A_L + d)g\big) - \frac{1}{12\pi}\mathrm{tr}[g^{-1}dg]^3 \tag{32}$$

$$+ \frac{1}{4\pi}\mathrm{tr}[(g^{-1}D_A g)F_R - (gD_A g^{-1})F_L] + \frac{1}{4\pi}d\mathrm{tr}[A_L g A_R g^{-1}]. \tag{33}$$

1.2 On Dividing by $U(1)_{\mathrm{diag}}$

Reserved.

References

[1] Yuya Tanizaki and Tin Sulejmanpasic, *Anomaly and global inconsistency matching: θ-angles, $SU(3)/U(1)^2$ nonlinear sigma model, $SU(3)$ chains and its generalizations*, arXiv:1805.11423.