Goursat's Lemma

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For Goursat's lemma, see [1] for a detailed discussion. We write the identity element of a group as e. In general, for a group homomorphism $f : A \to B$ and a subgroup $S \subset B$, note that $f^{-1}(S) = \{a \in A | f(a) \subset S\}$ is a group. Indeed, if $a, a' \in f^{-1}(S)$, then $f(aa') = f(a)f(a') \in S$, so $aa' \in f^{-1}(S)$.

Lemma 0.1 (Goursat)**.** *Let A, B be groups. There is a one-to-one correspondence between subgroups $C \subset A \times B$ and quintuples $Q = (\bar{A}, N_A, \bar{B}, N_B, \theta)$, where $N_A \lhd \bar{A} \subset A$, $N_B \lhd \bar{B} \subset B$, and θ is an isomorphism $\theta : \bar{A}/N_A \xrightarrow{\cong} \bar{B}/N_B$.*

(Proof [1]) Let C be a subgroup $C \subset A \times B$. Define

$$\bar{A} = \{a \in A | (a, b) \in C\}, \quad N_A = \{a \in A | (a, e) \in C\}, \tag{0.1}$$

$$\bar{B} = \{b \in B | (a, b) \in C\}, \quad N_B = \{b \in B | (e, b) \in C\}. \tag{0.2}$$

Since C is a group, $\bar{A}$ and $\bar{B}$ are also groups. For any $a' \in N_A$ and $a \in \bar{A}$, there exists $b \in B$ such that

$$(a, b)(a', e)(a, b)^{-1} = (aa'a^{-1}, e), \tag{0.3}$$

and therefore $aN_Aa^{-1} = N_A$, so $N_A \lhd \bar{A}$. Similarly $N_B \lhd \bar{B}$. Define θ by

$$\theta(aN_A) := bN_B, \quad \text{where } (a, b) \in C. \tag{0.4}$$

The right-hand side is well-defined: if $(a, b), (a, b') \in C$, then $(a, b)^{-1}(a, b') = (1, b^{-1}b') \in C$, hence $b^{-1}b' \in N_B$, i.e. $b \in b'N_B$. Similarly, if $(a, b), (a', b) \in C$, then $a \in a'N_A$, so the left-hand side is well-defined. The homomorphism property follows from $(a, b), (a', b') \in C$ and hence $C \ni (a, b)(a', b') = (aa', bb')$. If $\theta(a) = N_B$, then $(a, b) \in C$ and $(e, b) \in C$. Thus $(a, e) = (a, b)(e, b)^{-1} \in C$, so $a \in N_A$; hence θ is injective. For every $b \in \bar{B}$, there exists a such that $(a, b) \in C$, so θ is surjective. This constructs the quintuple $f(C) := (\bar{A}, N_A, \bar{B}, N_B, \theta)$.

Conversely, given a quintuple $Q = (\bar{A}, N_A, \bar{B}, N_B, \theta)$, the map θ defines the subset

$$\mathcal{G}_\theta := \{(aN_A, \theta(aN_A)) | aN_A \in \bar{A}/N_A\} \subset \bar{A}/N_A \times \bar{B}/N_B. \tag{0.5}$$

Since θ is a group homomorphism, $\mathcal{G}_\theta$ is a group. Taking the inverse image under the natural surjection

$$p : \bar{A} \times \bar{B} \to \bar{A}/N_A \times \bar{B}/N_B, \tag{0.6}$$

we obtain the subgroup

$$g(Q) := p^{-1}(\mathcal{G}_\theta) \subset \bar{A} \times \bar{B} \subset A \times B. \tag{0.7}$$

It remains to show that the correspondences f and g are inverse to one another.

First show $g \circ f = \mathrm{id}$. Let $C \subset A \times B$ be a subgroup, and let $f(C) = (\bar{A}, N_A, \bar{B}, N_B, \theta)$ be defined as above. Then

$$g(f(C)) = \{\, (a, b) \in A \times B \mid (aN_A,\ bN_B) \in \mathcal{G}_\theta \,\}. \tag{0.8}$$

By the definition of $\mathcal{G}_\theta$,

$$(aN_A,\ bN_B) \in \mathcal{G}_\theta \iff (a,b) \in C, \tag{0.9}$$

and hence $g(f(C)) = C$.

Next show $f \circ g = \mathrm{id}$. Let $Q = (\bar{A}, N_A, \bar{B}, N_B, \theta)$ be given, and set $C' := g(Q) = p^{-1}(\mathcal{G}_\theta) \subset \bar{A} \times \bar{B}$. Construct $f(C') = (\bar{A}', N'_A, \bar{B}', N'_B, \theta')$ from C'. We first note that

$$\bar{A}' = \{\, a \in A \mid (a,b) \in C' \,\} = \{\, a \in A \mid a \in \bar{A} \,\} = \bar{A}, \tag{0.10}$$

$$\bar{B}' = \{\, b \in B \mid (a,b) \in C' \,\} = \{\, b \in B \mid bN_B \in \theta(\bar{A}/N_A) = \bar{B}/N_B \,\} = \{\, b \in B \mid b \in \bar{B} \,\} = \bar{B}, \tag{0.11}$$

$$N'_A = \{\, a \in A \mid (a,e) \in C' \,\} = \{\, a \in A \mid \theta(aN_A) = N_B \,\} = \{\, a \in A \mid a \in N_A \,\} = N_A, \tag{0.12}$$

$$N'_B = \{\, b \in B \mid (e,b) \in C' \,\} = \{\, b \in B \mid bN_B = \theta(N_A) = N_B \,\} = \{\, b \in B \mid b \in N_B \,\} = N_B. \tag{0.13}$$

Finally, we show $\theta' = \theta$. By definition,

$$\theta'(aN_A) = bN_B, \quad \text{where } (a,b) \in C'. \tag{0.14}$$

But $(a,b) \in C'$ is equivalent to $a \in \bar{A}$ and $bN_B = \theta(aN_A)$, and therefore $\theta' = \theta$. □

References

[1] Kristine Bauer, Debasis Sen, Peter Zvengrowski, *A Generalized Goursat Lemma*, arXiv:1109.0024.