Memo: Higher Berry Curvature and Lattice Models

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Abstract

This is a memo on lattice models in Ref. [1] for which higher Berry curvature has a nonzero integral.

Zero Dimensions

As a family of Hamiltonians over the sphere S^2 whose ground-state Chern number is 1, take the unit vector $\boldsymbol{n} \in \mathbb{R}^3$, $\boldsymbol{n}^2 = 1$, and set

$$H_{0\mathrm{D}}(\boldsymbol{n}) = \boldsymbol{n} \cdot \boldsymbol{\sigma}. \tag{1}$$

Here $\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ are Pauli matrices.

One Dimension

At each site $j \in \mathbb{Z}$, put two spin-1/2 degrees of freedom, denoted by spin operators $\boldsymbol{\sigma}_j^L$ and $\boldsymbol{\sigma}_j^R$. Use

$$(\boldsymbol{n}, n_4) \in S^3, \quad \boldsymbol{n}^2 + n_4^2 = 1 \tag{2}$$

as coordinates on S^3, and define

$$H_{1\mathrm{D}}(\boldsymbol{n}, n_4) = \sum_{j \in \mathbb{Z}} \boldsymbol{n} \cdot (-\boldsymbol{\sigma}_j^L + \boldsymbol{\sigma}_j^R) \tag{3}$$

$$+ |n_4| \sum_{j \in \mathbb{Z}} \begin{cases} \boldsymbol{\sigma}_j^L \cdot \boldsymbol{\sigma}_j^R & (n_4 \leq 0), \\ \boldsymbol{\sigma}_j^R \cdot \boldsymbol{\sigma}_{j+1}^L & (n_4 \geq 0). \end{cases} \tag{4}$$

This model has Berry-curvature integral

$$\nu = \frac{1}{2\pi} \sum_{\Delta^3 \in S^3} F^{(3)}(\Delta^3) = 1. \tag{5}$$

- In (4), the first term must be chosen with opposite signs so that it is compatible with the antiferromagnetic interaction in the second term. Indeed, if one instead takes

$$H_{1\mathrm{D}}(\boldsymbol{n}, n_4) = \sum_{j \in \mathbb{Z}} \boldsymbol{n} \cdot (\boldsymbol{\sigma}_j^L + \boldsymbol{\sigma}_j^R) + |n_4| \sum_{j \in \mathbb{Z}} \begin{cases} \boldsymbol{\sigma}_j^L \cdot \boldsymbol{\sigma}_j^R & (n_4 \leq 0), \\ \boldsymbol{\sigma}_j^R \cdot \boldsymbol{\sigma}_{j+1}^L & (n_4 \geq 0), \end{cases} \tag{6}$$

  then the gap closes somewhere on S^3.

- There is no need to restrict to spin 1/2. For a general spin S, the same model as (4) gives a model with topological number $\nu = 2S$.

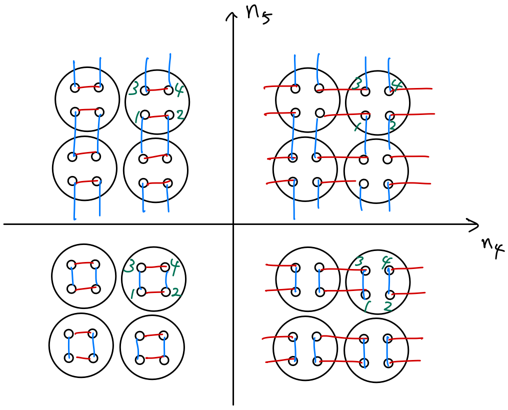

Figure 1: Two-dimensional model.

Two Dimensions

The two-dimensional model is similar. At each site, take four spin-1/2 degrees of freedom with Pauli matrices $\boldsymbol{\sigma}_j^1, \boldsymbol{\sigma}_j^2, \boldsymbol{\sigma}_j^3, \boldsymbol{\sigma}_j^4$, $j \in \mathbb{Z}^2$. Use

$$(\boldsymbol{n}, n_4, n_5) \in S^4, \quad \boldsymbol{n}^2 + n_4^2 + n_5^2 = 1 \tag{7}$$

as coordinates on S^4, and define

$$H_{\mathrm{2D}}(\boldsymbol{n}, n_4, n_5) = \sum_{j \in \mathbb{Z}^2} \boldsymbol{n} \cdot (-\boldsymbol{\sigma}_j^1 + \boldsymbol{\sigma}_j^2 + \boldsymbol{\sigma}_j^3 - \boldsymbol{\sigma}_j^4) \tag{8}$$

$$+ |n_4| \sum_{j \in \mathbb{Z}^2} \begin{cases} \boldsymbol{\sigma}_j^1 \cdot \boldsymbol{\sigma}_j^2 + \boldsymbol{\sigma}_j^3 \cdot \boldsymbol{\sigma}_j^4 & (n_4 \leq 0), \\ \boldsymbol{\sigma}_j^2 \cdot \boldsymbol{\sigma}_{j+\hat{x}}^1 + \boldsymbol{\sigma}_j^4 \cdot \boldsymbol{\sigma}_{j+\hat{x}}^3 & (n_4 \geq 0), \end{cases} \tag{9}$$

$$+ |n_5| \sum_{j \in \mathbb{Z}^2} \begin{cases} \boldsymbol{\sigma}_j^1 \cdot \boldsymbol{\sigma}_j^3 + \boldsymbol{\sigma}_j^2 \cdot \boldsymbol{\sigma}_j^4 & (n_5 \leq 0), \\ \boldsymbol{\sigma}_j^3 \cdot \boldsymbol{\sigma}_{j+\hat{y}}^1 + \boldsymbol{\sigma}_j^4 \cdot \boldsymbol{\sigma}_{j+\hat{y}}^2 & (n_5 \geq 0). \end{cases} \tag{10}$$

The second term is illustrated in Fig. 1. At any parameter point $(\boldsymbol{n}, n_4, n_5)$, this model reduces to a problem of four spin-1/2 degrees of freedom, and therefore can be solved easily. In particular, the ground state is unique and has a finite energy gap. The integral of the Berry curvature is again

$$\nu = \sum_{\Delta^4 \in S^4} F^{(4)}(\Delta^4) = 1. \tag{11}$$

General Spatial Dimension

The generalization to spatial dimensions three and higher should be straightforward. For example, in three spatial dimensions take $2^3 = 8$ spin-1/2 degrees of freedom at each site and define

$$H_{d\mathrm{D}}(\boldsymbol{n}, n_4, n_5, n_6) = \sum_{j\in\mathbb{Z}^3} \boldsymbol{n}\cdot(-\boldsymbol{\sigma}_j^1 + \boldsymbol{\sigma}_j^2 + \boldsymbol{\sigma}_j^3 - \boldsymbol{\sigma}_j^4 + \boldsymbol{\sigma}_j^5 - \boldsymbol{\sigma}_j^6 - \boldsymbol{\sigma}_j^7 + \boldsymbol{\sigma}_j^8) \tag{12}$$

$$+ |n_4| \sum_{j\in\mathbb{Z}^3} \begin{cases} \boldsymbol{\sigma}_j^1\cdot\boldsymbol{\sigma}_j^2 + \boldsymbol{\sigma}_j^3\cdot\boldsymbol{\sigma}_j^4 + \boldsymbol{\sigma}_j^5\cdot\boldsymbol{\sigma}_j^6 + \boldsymbol{\sigma}_j^7\cdot\boldsymbol{\sigma}_j^8 & (n_4 \le 0), \\ \boldsymbol{\sigma}_j^2\cdot\boldsymbol{\sigma}_{j+\hat{x}}^1 + \boldsymbol{\sigma}_j^4\cdot\boldsymbol{\sigma}_{j+\hat{x}}^3 + \boldsymbol{\sigma}_j^6\cdot\boldsymbol{\sigma}_{j+\hat{x}}^5 + \boldsymbol{\sigma}_j^8\cdot\boldsymbol{\sigma}_{j+\hat{x}}^7 & (n_4 \ge 0), \end{cases} \tag{13}$$

$$+ |n_5| \sum_{j\in\mathbb{Z}^3} \begin{cases} \boldsymbol{\sigma}_j^1\cdot\boldsymbol{\sigma}_j^3 + \boldsymbol{\sigma}_j^2\cdot\boldsymbol{\sigma}_j^4 + \boldsymbol{\sigma}_j^5\cdot\boldsymbol{\sigma}_j^7 + \boldsymbol{\sigma}_j^6\cdot\boldsymbol{\sigma}_j^8 & (n_5 \le 0), \\ \boldsymbol{\sigma}_j^3\cdot\boldsymbol{\sigma}_{j+\hat{y}}^1 + \boldsymbol{\sigma}_j^4\cdot\boldsymbol{\sigma}_{j+\hat{y}}^2 + \boldsymbol{\sigma}_j^7\cdot\boldsymbol{\sigma}_{j+\hat{y}}^5 + \boldsymbol{\sigma}_j^8\cdot\boldsymbol{\sigma}_{j+\hat{y}}^6 & (n_5 \ge 0), \end{cases} \tag{14}$$

$$+ |n_6| \sum_{j\in\mathbb{Z}^3} \begin{cases} \boldsymbol{\sigma}_j^1\cdot\boldsymbol{\sigma}_j^5 + \boldsymbol{\sigma}_j^2\cdot\boldsymbol{\sigma}_j^6 + \boldsymbol{\sigma}_j^3\cdot\boldsymbol{\sigma}_j^7 + \boldsymbol{\sigma}_j^4\cdot\boldsymbol{\sigma}_j^8 & (n_6 \le 0), \\ \boldsymbol{\sigma}_j^5\cdot\boldsymbol{\sigma}_{j+\hat{z}}^1 + \boldsymbol{\sigma}_j^6\cdot\boldsymbol{\sigma}_{j+\hat{z}}^2 + \boldsymbol{\sigma}_j^7\cdot\boldsymbol{\sigma}_{j+\hat{z}}^3 + \boldsymbol{\sigma}_j^8\cdot\boldsymbol{\sigma}_{j+\hat{z}}^4 & (n_6 \ge 0). \end{cases} \tag{15}$$

References

[1] Xueda Wen, Marvin Qi, Agnes Beaudry, Juan Moreno, Markus J. Pflaum, Daniel Spiegel, Ashvin Vishwanath, and Michael Hermele, *Flow of (higher) Berry curvature and bulk-boundary correspondence in parametrized quantum systems*, arXiv:2112.07748.