Line Gaps and Hermitianization

Ken Shiozaki

May 31, 2026

Following Ref. [1], we check that when a Hamiltonian $H(\boldsymbol{k})$ has a real line gap, it can be continuously deformed to a Hermitian Hamiltonian without closing the real line gap [2]. The following is essentially a copy of Appendix D of Ref. [1].

First, ignoring symmetry, we show that a Hamiltonian $H(\boldsymbol{k})$ with a real line gap can be flattened. The real-line-gap condition means that every eigenvalue $\epsilon_n(\boldsymbol{k})$ of $H(\boldsymbol{k})$ satisfies $|\mathrm{Re}\,\epsilon_n(\boldsymbol{k})| > 0$.[1] Introduce the continuous deformation

$$H_{\boldsymbol{k}}(\lambda) := (1-\lambda)H(\boldsymbol{k}) + \lambda \left(\oint_{C_+} - \oint_{C_-} \right) \frac{dz}{2\pi i} \frac{1}{z - H(\boldsymbol{k})}. \tag{1}$$

Here C_+ and C_- enclose the eigenvalues with $\mathrm{Re}\,\epsilon_n(\boldsymbol{k}) > 0$ and $\mathrm{Re}\,\epsilon_n(\boldsymbol{k}) < 0$, respectively. Since

$$P_\pm(\boldsymbol{k}) = \oint_{C_\pm} \frac{dz}{2\pi i} \frac{1}{z - H_{\boldsymbol{k}}} \tag{2}$$

satisfy $P_\pm(\boldsymbol{k})^2 = P_\pm(\boldsymbol{k})$,[2] they are projections. The eigenvalues of $H_\lambda(\boldsymbol{k})$ are

$$(1-\lambda)\epsilon_n(\boldsymbol{k}) + \lambda\, \mathrm{Re}\,\epsilon_n(\boldsymbol{k}), \tag{4}$$

so the real line gap is preserved for $\lambda \in [0,1]$. The eigenvalues of

$$H_1(\boldsymbol{k}) = P_+(\boldsymbol{k}) - P_-(\boldsymbol{k}) \tag{5}$$

are ±1. Equivalently, $H_1(\boldsymbol{k})^2 = 1$.

Next we Hermitianize it. Decompose $H_1(\boldsymbol{k})$ into real and imaginary Hermitian parts:

$$H_1(\boldsymbol{k}) = h_1(\boldsymbol{k}) + i h_2(\boldsymbol{k}), \tag{6}$$

$$h_1(\boldsymbol{k}) = \frac{H_1(\boldsymbol{k}) + H_1(\boldsymbol{k})^\dagger}{2}, \quad h_2(\boldsymbol{k}) = \frac{H_1(\boldsymbol{k}) - H_1(\boldsymbol{k})^\dagger}{2i}. \tag{7}$$

From $1 = H_1(\boldsymbol{k})^2 = h_1(\boldsymbol{k})^2 - h_2(\boldsymbol{k})^2 + i\{h_1(\boldsymbol{k}), h_2(\boldsymbol{k})\}$, we have

$$h_1(\boldsymbol{k})^2 - h_2(\boldsymbol{k})^2 = 1, \quad \{h_1(\boldsymbol{k}), h_2(\boldsymbol{k})\} = 0. \tag{8}$$

Introduce

$$\tilde{H}_\lambda(\boldsymbol{k}) = (1-\lambda)H_1(\boldsymbol{k}) + \lambda h_1(\boldsymbol{k}) = h_1(\boldsymbol{k}) + i(1-\lambda)h_2(\boldsymbol{k}). \tag{9}$$

Then

$$\tilde{H}_\lambda(\boldsymbol{k})^2 = h_1(\boldsymbol{k})^2 - (1-\lambda)^2 h_2(\boldsymbol{k})^2 = 1 + \big(1 - (1-\lambda)^2\big) h_2(\boldsymbol{k})^2. \tag{10}$$

---

[1] Here λ is an eigenvalue of a matrix A if there exists $u \neq 0$ such that $Au = \lambda u$, equivalently if $\lambda - A$ is singular.

[2] For example, see Ref. [3]; one uses the resolvent identity

$$(A-w)^{-1} - (A-z)^{-1} = (z-w)(A-z)^{-1}(A-w)^{-1} \tag{3}$$

and deforms the integration contours.

Because $h_2(\boldsymbol{k})$ is Hermitian, $\tilde{H}_\lambda(\boldsymbol{k})^2 \geq 1$. Thus the eigenvalues of $\tilde{H}_\lambda(\boldsymbol{k})$ are real and have nonzero absolute value. Hence $\tilde{H}_\lambda(\boldsymbol{k})$ preserves the real line gap for $\lambda \in [0,1]$. Since $\tilde{H}_1(\boldsymbol{k})$ is Hermitian, this gives the desired Hermitianization. Flattening $\tilde{H}_1(\boldsymbol{k})$ once more gives a flattened Hermitian Hamiltonian.

We now check compatibility with symmetry. We consider symmetries of the form

$$U_g(\boldsymbol{k}) \begin{cases} H(\boldsymbol{k}) \\ H(\boldsymbol{k})^* \\ H(\boldsymbol{k})^T \\ H(\boldsymbol{k})^\dagger \end{cases} U_g(\boldsymbol{k})^\dagger = c_g H(g\boldsymbol{k}), \tag{11}$$

where $c_g \in \{\pm 1\}$ and $U_g(\boldsymbol{k})$ is unitary. Taking the Hermitian conjugate of both sides shows that $H(\boldsymbol{k})^\dagger$ satisfies the same symmetry. Therefore $\tilde{H}_\lambda(\boldsymbol{k})$ satisfies the same symmetry whenever $H_1(\boldsymbol{k})$ does.

The flattened deformation $H_\lambda(\boldsymbol{k})$ also satisfies the same symmetry. First note that transposition does not affect the integration contour. When $c_g = -1$,

$$\left(\oint_{C_+} - \oint_{C_-}\right) \frac{dz}{2\pi i} \frac{1}{z - c_g H(g\boldsymbol{k})} = \left(\oint_{C_+} - \oint_{C_-}\right) \frac{c_g\, dz}{2\pi i} \frac{1}{c_g z - H(g\boldsymbol{k})}. \tag{12}$$

Changing variables $z \mapsto -z$ and using $C_\pm = -C_\mp$, one obtains

$$= c_g \left(\oint_{C_+} - \oint_{C_-}\right) \frac{dz}{2\pi i} \frac{1}{z - H(g\boldsymbol{k})}. \tag{13}$$

When complex conjugation is involved, one obtains

$$\left(\oint_{C_+} - \oint_{C_-}\right) \frac{dz}{-2\pi i} \frac{1}{z^* - H(g\boldsymbol{k})}. \tag{14}$$

Changing variables $z \mapsto z^*$ and using $C_\pm = -C_\pm$, this becomes

$$= \left(\oint_{C_+} - \oint_{C_-}\right) \frac{dz}{2\pi i} \frac{1}{z - H(g\boldsymbol{k})}. \tag{15}$$

Thus, under the symmetry (11), a Hamiltonian with a real line gap can be flattened and Hermitianized while preserving the symmetry.

If $H(\boldsymbol{k})$ has an imaginary line gap, then $H'(\boldsymbol{k}) := iH(\boldsymbol{k})$ has a real line gap, so the argument above Hermitianizes and flattens $H'(\boldsymbol{k})$. In addition to the transformation $H'(\boldsymbol{k}) = iH(\boldsymbol{k})$, note that for symmetries involving complex conjugation one has $c_g \mapsto -c_g$.

References

[1] Yuto Ashida, Zongping Gong, and Masahito Ueda, arXiv:2006.01837.

[2] Kohei Kawabata, Ken Shiozaki, Masahito Ueda, and Masatoshi Sato, arXiv:1812.09133.

[3] Tosio Kato, *A Short Introduction to Perturbation Theory for Linear Operators*, Springer, 2012.