On induced representations and irreducible decompositions

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June 1, 2026

Let G be a finite group, and let

$$\phi : G \to \{1, -1\}$$

specify whether a group element $g \in G$ is unitary or antiunitary. We denote the unitary subgroup by a subscript,

$$G_0 = \{g \in G | \phi_g = 1\}.$$

Introduce the abbreviations

$$x^{\phi_g} = \left\{ \begin{array}{ll} x & (\phi_g = 1) \\ x^* & (\phi_g = -1) \end{array} \right. , \quad x^{-\phi_g} = \left\{ \begin{array}{ll} x^* & (\phi_g = 1) \\ x & (\phi_g = -1). \end{array} \right. \tag{1}$$

We write representations and irreducible representations as $\alpha, \beta, \dots$. We write their representation matrices as D^α_g and their characters as χ^α_g. Equivalence of representations is denoted by $\alpha \sim \beta$, and $\alpha \sim \beta$ is equivalent to $\chi^\alpha_g = \chi^\beta_g$.

Write the multiplier as $z_{g,h} \in U(1)$, so that $\hat{g}\hat{h} = z_{g,h}\widehat{gh}$ for $g, h \in G$. It satisfies the 2-cocycle condition

$$z^{\phi_g}_{h,k} z^{-1}_{gh,k} z_{g,hk} z^{-1}_{g,h} = 1. \tag{2}$$

- Mapping representations.

  Let $H \subset G$. For a representation α of H_0, let the representation basis be $\{i\}_{i=1,\dots,\dim\alpha}$.

$$\hat{g} |j\rangle = \sum_{i=1}^{\dim\alpha} |i\rangle [D^\alpha_g]_{ij}, \quad g \in H_0. \tag{3}$$

  For $h \in G$, formally take the representation basis of the representation $h\alpha$ on hH_0h^{-1}, mapped by h, to be $\{\hat{h} |i\rangle\}_{i=1,\dots,\dim\alpha}$. The representation matrix is obtained from

$$\hat{g}\hat{h} |j\rangle = \frac{z_{g,h}}{z_{h,h^{-1}gh}} \hat{h}\widehat{h^{-1}gh} |j\rangle = \frac{z_{g,h}}{z_{h,h^{-1}gh}} \hat{h} \sum_{i=1}^{\dim\alpha} |i\rangle [D^\alpha_{h^{-1}gh}]_{ij}, \quad g \in hH_0h^{-1}, \tag{4}$$

  and hence

$$D^{h\alpha}_{g\in hH_0h^{-1}} = \frac{z_{g,h}}{z_{h,h^{-1}gh}} [D^\alpha_{h^{-1}gh}]^{\phi_h}. \tag{5}$$

  In particular, we obtain the formula for the mapped character

$$\boxed{\chi^{h\alpha}_{g\in hH_0h^{-1}} = \frac{z_{g,h}}{z_{h,h^{-1}gh}} [\chi^\alpha_{h^{-1}gh}]^{\phi_h},} \tag{6}$$

or, equivalently,

$$\boxed{\chi^{h\alpha}_{hgh^{-1}} = \frac{z_{hgh^{-1},h}}{z_{h,g}}[\chi^{\alpha}_{g}]^{\phi_h}, \quad g \in H_0.} \tag{7}$$

In particular, if the representation α of H_0 is mapped by $k \in H_0$, then $k\alpha$ is equivalent to α.[1] Therefore

$$\boxed{\chi^{k\alpha}_{g} = \frac{z_{g,k}}{z_{k,k^{-1}gk}}\chi^{\alpha}_{k^{-1}gk} = \chi^{\alpha}_{g}, \quad g,k \in H_0} \tag{8}$$

holds.

- The map of representations preserves the group structure.

  Let α be a representation of H_0. The following holds:

$$\boxed{h(k\alpha) = (hk)\alpha, \quad h,k \in G.} \tag{9}$$

Indeed,

$$\chi^{h(k\alpha)}_{g} = z_{g,h}z^{-1}_{h,h^{-1}gh}(\chi^{k\alpha}_{h^{-1}gh})^{\phi_h} = z_{g,h}z^{-1}_{h,h^{-1}gh}(z_{h^{-1}gh,k}z^{-1}_{k,k^{-1}h^{-1}ghk}(\chi^{\alpha}_{k^{-1}h^{-1}ghk})^{\phi_k})^{\phi_h} \tag{10}$$

$$= z_{g,h}z^{-1}_{h,h^{-1}gh}z^{\phi_h}_{h^{-1}gh,k}z^{-\phi_h}_{k,(hk)^{-1}ghk}(\chi^{\alpha}_{(hk)^{-1}ghk})^{\phi_{hk}}. \tag{11}$$

Using the 2-cocycle condition

$$z^{\phi_h}_{h^{-1}gh,k}z^{-1}_{gh,k}z_{h,h^{-1}ghk}z^{-1}_{h,h^{-1}gh} = 1, \tag{12}$$

we get

$$\chi^{h(k\alpha)}_{g} = z_{gh,k}z^{-1}_{h,h^{-1}ghk}z_{g,h}z^{-\phi_h}_{k,(hk)^{-1}ghk}(\chi^{\alpha}_{(hk)^{-1}ghk})^{\phi_{hk}}. \tag{13}$$

Furthermore, the 2-cocycle condition

$$z^{\phi_h}_{k,(hk)^{-1}ghk}z^{-1}_{hk,(hk)^{-1}ghk}z_{h,h^{-1}ghk}z^{-1}_{h,k} = 1 \tag{14}$$

gives

$$\chi^{h(k\alpha)}_{g} = z^{-1}_{hk,(hk)^{-1}ghk}z^{-1}_{h,k}z_{gh,k}z_{g,h}(\chi^{\alpha}_{(hk)^{-1}ghk})^{\phi_{hk}}. \tag{15}$$

Finally, noting that $\phi_g = 1$ and using the 2-cocycle condition

$$z_{h,k}z^{-1}_{gh,k}z_{g,hk}z^{-1}_{g,h} = 1, \tag{16}$$

we obtain

$$\chi^{h(k\alpha)}_{g} = z^{-1}_{hk,(hk)^{-1}ghk}z_{g,hk}(\chi^{\alpha}_{(hk)^{-1}ghk})^{\phi_{hk}} = \chi^{(hk)\alpha}_{g}. \tag{17}$$

In particular, there is freedom in the choice of h of the form $h \mapsto hk$ with $k \in H_0$. Since $k\alpha \sim \alpha$ for $k \in H_0$, the mapped character is independent of the choice of representative $h \in G$. That is,

$$\boxed{hk\alpha = h\alpha, \quad k \in H_0} \tag{18}$$

holds.

[1] This follows immediately from $D^{\alpha}_{g\in G_0}D^{\alpha}_{k} = \frac{z_{g,k}}{z_{k,k^{-1}gk}}D^{\alpha}_{k}D^{\alpha}_{k^{-1}gk}$.

- The map of representations and the inner product.

  Define the inner product of representations α, β of H_0 by

$$\boxed{(\alpha, \beta) := \frac{1}{|H_0|} \sum_{g \in H_0} (\chi_g^\alpha)^* \chi_g^\beta \in \mathbb{Z}_{\geq 0}.} \tag{19}$$

  The map of representations by $h \in G$ commutes with this inner product:

$$\boxed{(h\alpha, h\beta) = (\alpha, \beta).} \tag{20}$$

  Indeed,

$$(h\alpha, h\beta) = \frac{1}{|H_0|} \sum_{g \in hH_0h^{-1}} (\chi_g^{h\alpha})^* \chi_g^{h\beta} \tag{21}$$

$$= \frac{1}{|H_0|} \sum_{g \in hH_0h^{-1}} \left(\frac{z_{g,h}}{z_{h,h^{-1}gh}} (\chi_{h^{-1}gh}^\alpha)^{\phi_h} \right)^* \frac{z_{g,h}}{z_{h,h^{-1}gh}} (\chi_{h^{-1}gh}^\beta)^{\phi_h} \tag{22}$$

$$= (\alpha, \beta)^{\phi_h} = (\alpha, \beta). \tag{23}$$

- Restriction of representations.

  For a representation α of G_0, denote its restriction to $H_0 \subset G_0$ by $\alpha|_{H_0}$.

- Commutativity of restriction and mapping of representations.

  Restriction of representations and mapping of representations commute:

$$\boxed{h(\alpha|_{H_0}) = (h\alpha)|_{hH_0h^{-1}}.} \tag{24}$$

  Indeed, the character of $h\alpha$ is

$$\chi_{g \in hH_0h^{-1}}^{h\alpha} = \frac{z_{g,h}}{z_{h,h^{-1}gh}} (\chi_{h^{-1}gh \in H_0}^\alpha)^{\phi_h}, \tag{25}$$

  and the right-hand side is precisely the map by $h \in G$ of the restriction $\alpha|_{H_0}$.

- Induced representations.

  Write the left coset decomposition of $H_0 \subset G$ as

$$G = \coprod_{a=1}^{|G/H_0|} h_a H_0. \tag{26}$$

  We may choose $h_1 = e$. For a representation α of H_0, formally write the induced representation to G as

$$\mathrm{Ind}\alpha = \bigoplus_{a=1}^{|G/H_0|} h_a \alpha. \tag{27}$$

  Notice that h_a may include antiunitary group elements. If the representation basis of α is $\{i\}_{i=1}^{\dim \alpha}$, then the representation basis of the induced representation is formally given by

$$\{\hat{h}_a |i\rangle\}_{i=1,\dots,\dim\alpha, a=1,\dots,|G/H_0|}.$$

  For $g \in G_0$, we have

$$\hat{g}\hat{h}_a |i\rangle = z_{g,h_a} \widehat{gh_a} |i\rangle. \tag{28}$$

Writing $gh_a = h_b g'$ with $g' \in H_0$, we get

$$\hat{g}\hat{h}_a |j\rangle = z_{g,h_a} \widehat{h_b g'} |j\rangle = \frac{z_{g,h_a}}{z_{h_b,g'}} \hat{h}_b \hat{g}' |j\rangle \tag{29}$$

$$= \frac{z_{g,h_a}}{z_{h_b,g'}} (\hat{h}_b |i\rangle)[D^\alpha_{g'}]^{\phi_{h_b}}_{ij} = \frac{z_{g,h_a}}{z_{h_b,h_b^{-1}gh_a}} (\hat{h}_b |i\rangle)[D^\alpha_{h_b^{-1}gh_a}]^{\phi_{h_b}}_{ij}. \tag{30}$$

Only the terms with $h_a = h_b$, namely $gh_a \in h_a H_0$, contribute to the character of the induced representation. Thus

$$\boxed{\chi^{\mathrm{Ind}\alpha}_{g\in G_0} = \sum_{a=1}^{|G/H_0|} \delta_{g\in h_a H_0 h_a^{-1}} \chi^{h_a\alpha}_g.} \tag{31}$$

Compute its inner product with a representation β of G_0. The contribution from $h_a\alpha$ is

$$\frac{1}{|G_0|} \sum_{g\in G_0} (\chi^\beta_g)^* \delta_{g\in h_a H_0 h_a^{-1}} \chi^{h_a\alpha}_g = \frac{1}{|G_0|} \sum_{g\in h_a H_0 h_a^{-1}} (\chi^\beta_g)^* \chi^{h_a\alpha}_g \tag{32}$$

$$= \frac{|H_0|}{|G_0|} (\beta|_{h_a H_0 h_a^{-1}}, h_a\alpha). \tag{33}$$

This can be written as the inner product with the restriction of β to H_0. Therefore the inner product between the induced representation and a representation of G_0 is

$$\boxed{(\beta, \mathrm{Ind}\alpha) = \frac{|H_0|}{|G_0|} \sum_{a=1}^{|G/H_0|} (\beta|_{h_a H_0 h_a^{-1}}, h_a\alpha),} \tag{34}$$

that is, it is given by the sum, over a, of the inner products between the restriction of β to the subgroup $h_a H_0 h_a^{-1}$ and $h_a\alpha$. Thus the irreducible decomposition of the induced representation can be computed once the decomposition matrices of irreducible representations β of G_0 with respect to irreducible representations $h_a\alpha$ of $h_a H_0 h_a^{-1}$ are known.

Furthermore, using the commutativity of the inner product with the map of representations, we may write

$$(\beta, \mathrm{Ind}\alpha) = \frac{|H_0|}{|G_0|} \sum_{a=1}^{|G/H_0|} ((h_a^{-1}\beta)|_{H_0}, \alpha). \tag{35}$$

If h_a is unitary, then $h_a\beta \sim \beta$, so $((h_a^{-1}\beta)|_{H_0}, \alpha) = (\beta|_{H_0}, \alpha)$ and the a-dependence disappears. On the other hand, if h_a is antiunitary, $h_a\beta$ and β may in general be inequivalent representations, but the result depends only on an antiunitary representative $T \in G$ with $\phi_T = -1$. Writing $G = G_0 \coprod TG_0$, we have

$$(\beta, \mathrm{Ind}\alpha) = \frac{|H_0|}{|G_0|} \Big(\sum_{h_a\in G_0} + \sum_{h_a\in TG_0} \Big) ((h_a^{-1}\beta)|_{H_0}, \alpha) \tag{36}$$

$$= (\beta|_{H_0}, \alpha) + ((T^{-1}\beta)|_{H_0}, \alpha). \tag{37}$$

Equivalently, noting that $T^{-1} = TT^{-2}$ and $T^{-2} \in G_0$, we obtain

$$\boxed{(\beta, \mathrm{Ind}\alpha) = ((\beta \oplus T\beta)|_{H_0}, \alpha).} \tag{38}$$

It can also be written as

$$\boxed{(\beta, \mathrm{Ind}\alpha) = (\beta|_{H_0}, \alpha) + (\beta|_{TH_0T^{-1}}, T\alpha).} \tag{39}$$

This is the relation known as Frobenius reciprocity.