

A Note on the Definition of the Relative K -Group

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I was told that it is incorrect to define the relative K -group $K^0(X, Y)$ using pairs $(H_0(x), H_1(x))$ with the condition

$$H_0(x) \text{ and } H_1(x) \text{ are homotopy equivalent on } Y. \quad (1)$$

This note records an example where this definition causes a problem.¹ One correct definition of the relative K -group seems to impose instead

$$H_0(x) = H_1(x) \text{ for } x \in Y. \quad (2)$$

As a fact, $K^0(D^2, \partial D^2) \cong \mathbb{Z}$. Put coordinates (r, θ) on D^2 , with $0 \leq r \leq 1$. Using the second definition, a generator is given by

$$H_0(r, \theta) = r \cos \theta \sigma_x + r \sin \theta \sigma_y + \sqrt{1 - r^2} \sigma_z, \quad (3)$$

$$H_1(r, \theta) = r \cos \theta \sigma_x + r \sin \theta \sigma_y - \sqrt{1 - r^2} \sigma_z. \quad (4)$$

If one adopts the first definition, however, the above $H_0(r, \theta)$ and $H_1(r, \theta)$ become homotopy equivalent on D^2 , as we now explain.

Writing $H = \mathbf{h} \cdot \boldsymbol{\sigma}$, we regard H as a point of the sphere S^2 via $\mathbf{h}/|\mathbf{h}| \in S^2$. Then the image of D^2 under $H_0(r, \theta)$ is the northern hemisphere, while the image under $H_1(r, \theta)$ is the southern hemisphere. On the boundary $\partial D^2 \cong S^1$, both maps run along the equator of S^2 .

Since S^1 does not wind around S^2 , any two maps $\partial D^2 \rightarrow S^2$ are homotopy equivalent. Keeping this point in mind, consider for example

$$H_0(r, \theta, t) = (1 - t)r \cos \theta \sigma_x + (1 - t)r \sin \theta \sigma_y + \sqrt{1 - (1 - t)^2 r^2} \sigma_z \xrightarrow{t \rightarrow 1} \sigma_z, \quad (5)$$

$$H_1(r, \theta, t) = (1 - t)r \cos \theta \sigma_x + (1 - t)r \sin \theta \sigma_y - \sqrt{1 - (1 - t)^2 r^2} \sigma_z \xrightarrow{t \rightarrow 1} -\sigma_z. \quad (6)$$

Thus $H_0(r, \theta)$ and $H_1(r, \theta)$ are homotopy equivalent to the constant maps σ_z and $-\sigma_z$, respectively. Since the two constant maps σ_z and $-\sigma_z$ are themselves homotopy equivalent, the first definition would make $H_0(r, \theta)$ and $H_1(r, \theta)$ homotopy equivalent. Consequently, this pair would not represent the nontrivial element of $K^0(D^2, \partial D^2)$.

¹I thank Mayuko Yamashita for pointing this out.