# A Calculation Note on the LSM Theorem

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**Abstract**

This is a note on the energy estimate used in the Lieb–Schultz–Mattis theorem.

Following Section 6.2 of [1], we track the calculation. Consider the Hamiltonian of a one-dimensional antiferromagnetic chain:

$$H = \sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}. \tag{0.1}$$

The spin magnitude $S$ is arbitrary. Let $|\psi\rangle$ be the ground state. Introduce the twist operator

$$U_{\rm tw} = e^{-i\sum_j \theta_j S^z_j}, \tag{0.2}$$

where

$$\theta_j = \begin{cases} 0 & (j<0), \\ \frac{2\pi j}{L} & (j=1,\dots,L), \\ 2\pi = 0 & (j \geq L). \end{cases} \tag{0.3}$$

We estimate the energy of the trial state

$$|\psi_{\rm tw}\rangle = U_{\rm tw}|\psi\rangle. \tag{0.4}$$

The energy difference is

$$\begin{aligned} \Delta E &= \langle \psi_{\rm tw}|H|\psi_{\rm tw}\rangle - E_0 \\ &= \langle \psi_{\rm tw}|H|\psi_{\rm tw}\rangle - \langle\psi|H|\psi\rangle = \langle\psi|U^\dagger_{\rm tw} H U_{\rm tw} - H|\psi\rangle. \end{aligned} \tag{0.5}$$

From only the spin commutation relations $[S^\mu, S^\nu] = i\epsilon_{\mu\nu\rho}S^\rho$, we have

$$e^{i\theta S^z} S^{\pm} e^{-i\theta S^z} = e^{\pm i\theta} S^{\pm}, \quad S^{\pm} = S^x \pm i S^y. \tag{0.6}$$

Therefore

$$\begin{aligned} &U^\dagger_{\rm tw} \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} U_{\rm tw} \\ &= U^\dagger_{\rm tw}\left(\frac{S^+_j S^-_{j+1} + S^-_j S^+_{j+1}}{2} + S^z_j S^z_{j+1}\right) U_{\rm tw} \\ &= \frac{e^{i(\theta_j - \theta_{j+1})} S^+_j S^-_{j+1} + e^{-i(\theta_j-\theta_{j+1})} S^-_j S^+_{j+1}}{2} + S^z_j S^z_{j+1} \\ &= \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{2}\Big[(e^{i(\theta_j-\theta_{j+1})} - 1) S^+_j S^-_{j+1} + (e^{-i(\theta_j-\theta_{j+1})} - 1) S^-_j S^+_{j+1}\Big]. \end{aligned} \tag{0.7}$$

Thus

$$\Delta E = \frac{1}{2} \sum_j \langle \psi | (e^{i(\theta_j - \theta_{j+1})} - 1) S_j^+ S_{j+1}^- + (e^{-i(\theta_j - \theta_{j+1})} - 1) S_j^- S_{j+1}^+ | \psi \rangle \,. \tag{0.8}$$

Following [1], we estimate the right-hand side of

$$\begin{aligned} \Delta E &= \langle \psi | U_{\mathrm{tw}}^\dagger H U_{\mathrm{tw}} - H | \psi \rangle \\ &\leq \langle \psi | U_{\mathrm{tw}}^\dagger H U_{\mathrm{tw}} - H | \psi \rangle + \langle \psi | U_{\mathrm{tw}} H U_{\mathrm{tw}}^\dagger - H | \psi \rangle \,. \end{aligned} \tag{0.9}$$

This gives

$$\begin{aligned} \Delta E &\leq \sum_j \langle \psi | (\cos(\theta_j - \theta_{j+1}) - 1) S_j^+ S_{j+1}^- + (\cos(\theta_j - \theta_{j+1}) - 1) S_j^- S_{j+1}^+ | \psi \rangle \\ &= \sum_{j=0}^{L-1} (\cos(\theta_j - \theta_{j+1}) - 1) \langle \psi | S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+ | \psi \rangle \\ &\leq \sum_{j=0}^{L-1} 2(1 - \cos(\theta_j - \theta_{j+1})) \left| \langle \psi | S_j^x S_{j+1}^x + S_j^y S_{j+1}^y | \psi \rangle \right| \\ &\leq \sum_{j=0}^{L-1} 2(1 - \cos(\theta_j - \theta_{j+1})) \| S_j^x S_{j+1}^x + S_j^y S_{j+1}^y \| \\ &\leq 2 \left( 1 - \cos \frac{2\pi}{L} \right) \times 2S^2 L \\ &\leq \left( \frac{2\pi}{L} \right)^2 2S^2 L = \frac{8\pi^2 S^2}{L}. \end{aligned} \tag{0.10}$$

# References


[1] Hal Tasaki, *Physics and Mathematics of Quantum Many-Body Systems*, Vol. 66, Springer, 2020.