A Lattice Approximation to the Three-Dimensional Winding Number

Ken Shiozaki

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Abstract

1 Winding Number and the WZW Term

Consider a map from the two-dimensional torus T^2 to $U(n)$,

$$g : T^2 \to U(n), \quad (k_x, k_y) \mapsto g(k_x, k_y). \tag{1}$$

If the one-dimensional winding number

$$W_1 = \frac{1}{2\pi i} \oint d \log \det g \tag{2}$$

vanishes, then there exists a homotopy $\tilde{g}(k_x, k_y, t \in [0,1])$ to a point $g_0 \in U(n)$:

$$\tilde{g}(k_x, k_y, 0) = g(k_x, k_y), \quad \tilde{g}(k_x, k_y, 1) = g_0. \tag{3}$$

In that case the WZW term is defined by

$$\Theta = \frac{1}{12\pi} \int_{T^2 \times [0,1]} \mathrm{tr}\, [\tilde{g}^{-1} d\tilde{g}]^3 \in \mathbb{R}/\mathbb{Z}. \tag{4}$$

Here

$$\omega = \frac{1}{24\pi^2} \mathrm{tr}\, [g^{-1} dg]^3 \tag{5}$$

is the volume element on $U(n)$, and for an oriented closed three-dimensional space M_3,

$$W_3 = \frac{1}{24\pi^2} \int_{M_3} \mathrm{tr}\, [g^{-1} dg]^3 \in \mathbb{Z} \tag{6}$$

is an integer. The purpose of this note is to give a way to compute the WZW term Θ by a lattice approximation of the torus T^2.

- When the one-dimensional winding number W_1 is nonzero, is the WZW term not definable? If the map can be extended to a three-dimensional Hamiltonian without changing the value of the WZW term, it seems that it should be definable as a CS3 form.

2 Constructing Singular Chains from Points in $U(n)$

2.1 Two points

In general, one path connecting two points g_0, g_1 in $U(n)$ can be constructed as follows. Diagonalize the $U(n)$ matrix

$$g_1 g_0^{-1}. \tag{7}$$

Write

$$g_1 g_0^{-1} = U\Lambda U^\dagger, \qquad \Lambda = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}). \tag{8}$$

If $e^{i\theta_j} \neq -1$ for every j, then

$$\mathrm{Arg}\,\Lambda = \mathrm{diag}(\theta_1, \ldots, \theta_n), \quad |\theta_j| < \pi, \tag{9}$$

is uniquely determined. Thus the Hermitian matrix

$$h_{01} = U(\mathrm{Arg}\,\Lambda)U^\dagger \tag{10}$$

is determined, and

$$g_1 g_0^{-1} = e^{ih_{01}}. \tag{11}$$

One path $g_{01}(t)$ connecting g_0 and g_1 is

$$g_{01}(t) = e^{ith_{01}} g_0, \quad t \in [0,1]. \tag{12}$$

Is the path $g_{10}(t)$ constructed after exchanging the order of g_0 and g_1 the same as $g_{01}(t)$? Since $g_0 g_1^{-1} = e^{ih_{10}} = e^{-ih_{01}}$, we have

$$g_{10}(t) = e^{-ith_{01}} g_1. \tag{13}$$

Then

$$\begin{aligned} g_{10}(1-t) &= e^{-i(1-t)h_{01}} g_1 = e^{ith_{01}} e^{-ih_{01}} g_1 \\ &= e^{ith_{01}} (g_1 g_0^{-1})^{-1} g_1 = e^{ith_{01}} g_0 = g_{01}(t), \end{aligned} \tag{14}$$

so it is the same path with the opposite orientation.

- Is $g_{01}(t)$ a geodesic?

2.2 Three points

One way to construct a singular two-chain $\Delta^2 \to U(n)$ with vertices g_0, g_1, g_2 is to give a path connecting g_0 and $g_{12}(t)$. Namely, introduce the Hermitian matrix $h_{012}(t)$ determined by

$$g_{12}(t) g_0^{-1} = e^{ith_{12}} g_1 g_0^{-1} = e^{ith_{12}} e^{ih_{01}} =: e^{ih_{012}(t)}, \tag{15}$$

and define

$$g_{012}(s,t) := e^{ish_{012}(t)} g_0, \quad 0 \leq s, t \leq 1. \tag{16}$$

Notice that

$$e^{ih_{012}(0)} = g_1 g_0^{-1} = e^{ih_{01}}, \quad e^{ih_{012}(1)} = g_2 g_0^{-1} = e^{ih_{02}}. \tag{17}$$

Also note that $g_2 g_1^{-1} g_1 g_0^{-1} = g_2 g_0^{-1}$, that is,

$$e^{ih_{12}} e^{ih_{01}} = e^{ih_{02}}. \tag{18}$$

We have

$$g_{012}(s,0) = e^{ish_{012}(0)} g_0 = g_{01}(s), \tag{19}$$

$$g_{012}(s,1) = e^{ish_{012}(1)} g_0 = g_{02}(s), \tag{20}$$

$$g_{012}(1,t) = e^{ih_{012}(t)} g_0 = g_{12}(t). \tag{21}$$

The singular two-chain $g_{012}(s,t)$ depends on the ordering of the vertices $0, 1, 2$.

2.3 Four points

Similarly, one singular three-chain $\Delta^3 \to U(n)$ with vertices g_0, g_1, g_2, g_3 is obtained by constructing a path connecting the two points g_0 and $g_{123}(s,t)$. Define the Hermitian matrix $h_{0123}(s,t)$ by

$$g_{123}(s,t)g_0^{-1} =: e^{ih_{0123}(s,t)}. \tag{22}$$

Define the singular three-chain by

$$g_{0123}(r,s,t) = e^{irh_{0123}(s,t)}g_0, \quad 0 \le r,s,t \le 1. \tag{23}$$

Again, $g_{0123}(r,s,t)$ depends on the ordering of the points g_0, g_1, g_2, g_3.

2.4 Winding number

For a map $g : M_3 \to U(n)$, we want to compute the winding number W_3 by a discrete approximation. Choose a triangulation $|M_3|$ of M_3 and fix a branching structure. Then for each three-cell $\Delta^3 = (0123)$ of $|M_3|$, the method of the previous subsection determines a map $\Delta^3 \to U(n)$. Since the branching structure has been fixed, the boundaries of adjacent three-cells are mapped to the same points of $U(n)$. Therefore the winding number W_3 is given by the sum of the integrals of the volume element ω over all three-cells:

$$W_3 = \sum_{\Delta^3 \in |M_3|} \sigma(\Delta^3) \int_{\Delta^3} \omega(g_{0123}), \tag{24}$$

$$\int_{\Delta^3} \omega(g_{0123}) := \frac{1}{24\pi^2} \int_{\Delta^3} \mathrm{tr}\,[g_{0123}^{-1} dg_{0123}]^3. \tag{25}$$

Here $\sigma(\Delta^3) \in \{\pm 1\}$ is the sign factor indicating whether the orientation of Δ^3 agrees with that of M_3.

3 Discrete Approximation of the Volume Element of $U(n)$

Can the integral $\int_{\Delta^3} \omega(g_{0123})$ be computed simply from the four vertices $g_0, g_1, g_2, g_3 \in U(n)$?

Alternatively, is there another discrete approximation of W_3 such that the sum

$$\sum_{\Delta^3} \sigma(\Delta^3) \int_{\Delta^3} \omega(g_{0123}) \tag{26}$$

always takes an integer value?

Introduce the Hermitian matrices h_{01}, h_{02}, h_{03} determined by

$$e^{ih_{0j}} = g_j g_0^{-1}, \quad j = 1,2,3. \tag{27}$$

Then one can approximate

$$\int_{\Delta^3} \omega(g_{0123}) \sim \frac{1}{24\pi^2} \frac{i}{2} \mathrm{tr}\,[h_{01}[h_{02}, h_{03}]]. \tag{28}$$

However, numerical calculations [1] show that as the number of discrete points L is increased, the approximation converges to W_3, but it is not quantized at each finite L. The result is shown in Fig. 1. Here $g(k_x, k_y, k_z)$ is obtained by unitarizing q via singular-value decomposition, where

$$q = \sin k_x \sigma_x + \sin k_y \sigma_y + \sin k_z \sigma_z + i(\cos k_x + \cos k_y + \cos k_z + m). \tag{29}$$

- One could also consider a method that discretizes $\mathrm{tr}\,[g^{-1}dg]^3$ naively and integrates it. How accurate would that be compared with the present method?

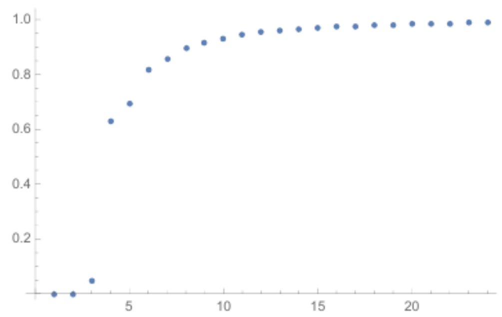

Figure 1: Result for $m = 2$. The number of lattice points is L^3; the horizontal axis is L. The vertical axis is the discrete approximation to the winding number W_3.

4 Computation Memo for the Case of $SU(2)$

We have

$$g = e^{i\theta \boldsymbol{n}\cdot\boldsymbol{\sigma}/2} = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\boldsymbol{n}\cdot\boldsymbol{\sigma}. \tag{30}$$

Since

$$\frac{1}{2}\mathrm{tr}\left[g\sigma_\mu\right] = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\boldsymbol{n}\right), \tag{31}$$

we get

$$\frac{\theta}{2} = \cos^{-1}\left(\frac{1}{2}\mathrm{tr}\left[g\right]\right), \tag{32}$$

$$\boldsymbol{n} = \frac{\frac{1}{2}\mathrm{tr}\left[g\boldsymbol{\sigma}\right]}{i\sin\frac{\theta}{2}} = \frac{\frac{1}{2}\mathrm{tr}\left[g\boldsymbol{\sigma}\right]}{i\sin\cos^{-1}\left(\frac{1}{2}\mathrm{tr}\left[g\right]\right)}. \tag{33}$$

Therefore the Hermitian matrix is given by

$$h = \frac{\theta}{2}\boldsymbol{n}\cdot\boldsymbol{\sigma} = \cos^{-1}\left(\frac{1}{2}\mathrm{tr}\left[g\right]\right)\frac{\frac{1}{2}\mathrm{tr}\left[g\boldsymbol{\sigma}\right]}{i\sin\cos^{-1}\left(\frac{1}{2}\mathrm{tr}\left[g\right]\right)}\cdot\boldsymbol{\sigma}. \tag{34}$$

References

[1] 3DWinding.nb.