Computation memo: Does the Chen–Gu–Liu–Wen cocycle model agree with the Levin–Gu model?

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Abstract

Even after using the degrees of freedom of coboundaries, I could not make them agree. The Levin–Gu model seems to be based on the string-net model, so the construction of the string-net model should be examined.

1 Levin–Gu model

Place spin-1/2 degrees of freedom on the vertices of the triangular lattice. The trivial Hamiltonian is

$$H_0 = -\sum_j \sigma_j^x. \tag{1}$$

The ground state is a superposition over arbitrary configurations of up and down spins:

$$|\Psi_0\rangle = \sum_\sigma |\sigma\rangle. \tag{2}$$

We do not keep track of the normalization factor. The ground state of the Levin–Gu model assigns a phase (-1) to each domain-wall loop [2]:

$$|\Psi_1\rangle = \sum_\sigma (-1)^{N_l} |\sigma\rangle. \tag{3}$$

Here N_l is the number of loops. The Levin–Gu ground state is obtained from the trivial state by applying the following local unitary transformation, namely finite-time evolution by a local Hamiltonian; see Appendix A of [2]:

$$U_\theta = \prod_{<pqr>} e^{i\theta(3\sigma_p^z\sigma_q^z\sigma_r^z - \sigma_p^z - \sigma_q^z - \sigma_r^z)}. \tag{4}$$

Here $<pqr>$ runs over all triangles. In particular, $\theta = \frac{\pi}{24}$ corresponds to the Levin–Gu model. (Check the $\mathbb{Z}_2$ symmetry.) Alternatively, it can also be written as follows [3]:

$$W_{\mathbb{Z}_2} = \prod_j e^{-i\frac{\pi}{12}\sigma_j^z \sum_{<ll'>}^j (\frac{1-\sigma_l^z\sigma_{l'}^z}{2})}. \tag{5}$$

Here $<ll'>$ runs over triangles that contain the site j. In [3] the coefficient is written as $e^{-i\frac{\pi}{6}\cdots}$, but this is probably a mistake. The Hamiltonian is obtained from

$$B_j = U_\theta \sigma_j^x U_\theta^{-1} = -\sigma_j^x e^{i\frac{\pi}{2}\sum_{pq}^j \frac{1-\sigma_p^z\sigma_q^z}{2}}, \tag{6}$$

as

$$H_1 = -\sum_j B_j. \tag{7}$$

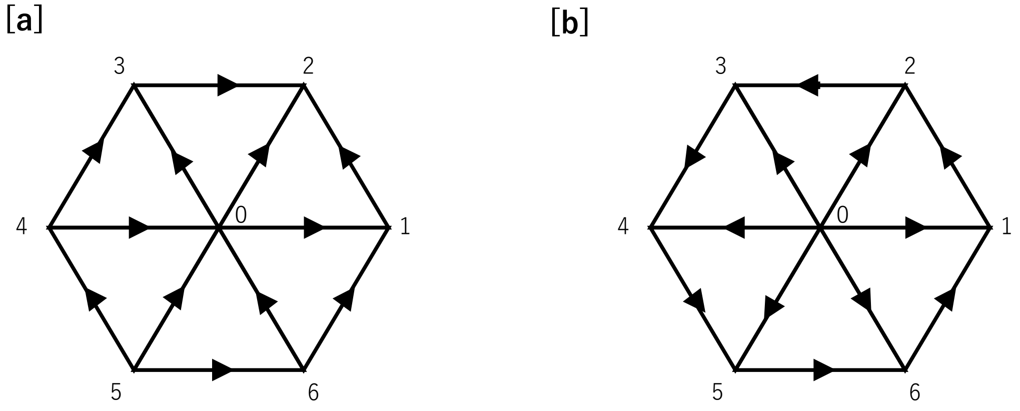

Figure 1

2 Cocycle model

Let $G = \mathbb{Z}_2 = \{0, 1\}$. Take the 3-cocycle

$$\omega(a, b, c) = (-1)^{abc}. \tag{8}$$

The homogeneous cocycle is

$$\nu(0, a, b, c) = \omega(a, -a + b, -b + c) = (-1)^{a(-a+b)(-b+c)} = \begin{cases} -1 & (a = 1, b = 0, c = 1), \\ 1 & \text{(else)}. \end{cases} \tag{9}$$

Here the reference group element was chosen to be $g_* = 0$. The unitary transformation in the cocycle model is [1]

$$U(\nu) = \sum_{\{a_j\}} \prod_{\Delta^2} \nu(0, a_0, a_1, a_2)^{\mathrm{sign}\Delta^2} \left|\{a_j\}\right\rangle \left\langle\{a_j\}\right|. \tag{10}$$

Noting that, on each 2-simplex,

$$\sum_{a_0, a_1, a_2} \nu(0, a_0, a_1, a_2) \left|a_0 a_1 a_2\right\rangle \left\langle a_0 a_1 a_2\right| = (-1)^{\frac{1-\sigma_0^z}{2} \frac{1+\sigma_1^z}{2} \frac{1-\sigma_2^z}{2}}, \tag{11}$$

we can write

$$U(\nu) = \prod_{\Delta^2} (-1)^{\frac{1-\sigma_0^z}{2} \frac{1+\sigma_1^z}{2} \frac{1-\sigma_2^z}{2}}. \tag{12}$$

Take the numbering of the sites near site 0 as shown in the figure. If the branching rule is chosen as in Fig. [a] so as to preserve translation symmetry, a direct calculation gives

$$
\begin{aligned}
&U(\nu)\sigma_0^x U(\nu)^{-1} \\
&= e^{\pi i\left[\frac{1-\sigma_0^z}{2}\frac{1+\sigma_1^z}{2}\frac{1-\sigma_2^z}{2}-\frac{1-\sigma_0^z}{2}\frac{1+\sigma_3^z}{2}\frac{1-\sigma_2^z}{2}+\frac{1-\sigma_4^z}{2}\frac{1+\sigma_0^z}{2}\frac{1-\sigma_3^z}{2}-\frac{1-\sigma_5^z}{2}\frac{1+\sigma_4^z}{2}\frac{1-\sigma_0^z}{2}+\frac{1-\sigma_5^z}{2}\frac{1+\sigma_6^z}{2}\frac{1-\sigma_0^z}{2}-\frac{1-\sigma_6^z}{2}\frac{1+\sigma_0^z}{2}\frac{1-\sigma_1^z}{2}\right]} \\
&\quad \sigma_0^x e^{-\pi i\left[\frac{1-\sigma_0^z}{2}\frac{1+\sigma_1^z}{2}\frac{1-\sigma_2^z}{2}-\frac{1-\sigma_0^z}{2}\frac{1+\sigma_3^z}{2}\frac{1-\sigma_2^z}{2}+\frac{1-\sigma_4^z}{2}\frac{1+\sigma_0^z}{2}\frac{1-\sigma_3^z}{2}-\frac{1-\sigma_5^z}{2}\frac{1+\sigma_4^z}{2}\frac{1-\sigma_0^z}{2}+\frac{1-\sigma_5^z}{2}\frac{1+\sigma_6^z}{2}\frac{1-\sigma_0^z}{2}-\frac{1-\sigma_6^z}{2}\frac{1+\sigma_0^z}{2}\frac{1-\sigma_1^z}{2}\right]} \\
&= \sigma_0^x \exp \pi i \Bigg[\frac{1+\sigma_0^z}{2}\frac{1+\sigma_1^z}{2}\frac{1-\sigma_2^z}{2}-\frac{1+\sigma_0^z}{2}\frac{1+\sigma_3^z}{2}\frac{1-\sigma_2^z}{2}+\frac{1-\sigma_4^z}{2}\frac{1-\sigma_0^z}{2}\frac{1-\sigma_3^z}{2} \\
&\qquad -\frac{1-\sigma_5^z}{2}\frac{1+\sigma_4^z}{2}\frac{1+\sigma_0^z}{2}+\frac{1-\sigma_5^z}{2}\frac{1+\sigma_6^z}{2}\frac{1+\sigma_0^z}{2}-\frac{1-\sigma_6^z}{2}\frac{1-\sigma_0^z}{2}\frac{1-\sigma_1^z}{2} \\
&\qquad -\frac{1-\sigma_0^z}{2}\frac{1+\sigma_1^z}{2}\frac{1-\sigma_2^z}{2}+\frac{1-\sigma_0^z}{2}\frac{1+\sigma_3^z}{2}\frac{1-\sigma_2^z}{2}-\frac{1-\sigma_4^z}{2}\frac{1+\sigma_0^z}{2}\frac{1-\sigma_3^z}{2} \\
&\qquad +\frac{1-\sigma_5^z}{2}\frac{1+\sigma_4^z}{2}\frac{1-\sigma_0^z}{2}-\frac{1-\sigma_5^z}{2}\frac{1+\sigma_6^z}{2}\frac{1-\sigma_0^z}{2}+\frac{1-\sigma_6^z}{2}\frac{1+\sigma_0^z}{2}\frac{1-\sigma_1^z}{2}\Bigg] \\
&= \sigma_0^x \exp \pi i \sigma_0^z \left[\frac{1+\sigma_1^z}{2}\frac{1-\sigma_2^z}{2}-\frac{1+\sigma_3^z}{2}\frac{1-\sigma_2^z}{2}-\frac{1-\sigma_4^z}{2}\frac{1-\sigma_3^z}{2}-\frac{1-\sigma_5^z}{2}\frac{1+\sigma_4^z}{2}+\frac{1-\sigma_5^z}{2}\frac{1+\sigma_6^z}{2}+\frac{1-\sigma_6^z}{2}\frac{1-\sigma_1^z}{2}\right] \\
&= \sigma_0^x \exp \pi i \left[\frac{1+\sigma_1^z}{2}\frac{1-\sigma_2^z}{2}-\frac{1+\sigma_3^z}{2}\frac{1-\sigma_2^z}{2}-\frac{1-\sigma_4^z}{2}\frac{1-\sigma_3^z}{2}-\frac{1-\sigma_5^z}{2}\frac{1+\sigma_4^z}{2}+\frac{1-\sigma_5^z}{2}\frac{1+\sigma_6^z}{2}+\frac{1-\sigma_6^z}{2}\frac{1-\sigma_1^z}{2}\right].
\end{aligned}
$$

Here we used the fact that the expression inside the brackets takes integer values and is independent of $\sigma_0^z = \pm 1$. Expanding the expression in brackets further, we obtain

$$
U(\nu)\sigma_0^x U(\nu)^{-1} = \sigma_0^x \exp \frac{\pi i}{2}\left[\frac{1-\sigma_1^z\sigma_2^z}{2}-\frac{1-\sigma_2^z\sigma_3^z}{2}-\frac{1+\sigma_3^z\sigma_4^z}{2}-\frac{1-\sigma_4^z\sigma_5^z}{2}+\frac{1-\sigma_5^z\sigma_6^z}{2}+\frac{1+\sigma_6^z\sigma_1^z}{2}\right]. \tag{13}
$$

This differs from the expression for the Levin–Gu model,

$$
B_0 = -\sigma_0^x i^{\frac{1-\sigma_1^z\sigma_2^z}{2}+\frac{1-\sigma_2^z\sigma_3^z}{2}+\frac{1-\sigma_3^z\sigma_4^z}{2}+\frac{1-\sigma_4^z\sigma_5^z}{2}+\frac{1-\sigma_5^z\sigma_6^z}{2}+\frac{1-\sigma_6^z\sigma_1^z}{2}}. \tag{14}
$$

It can also be written as

$$
\begin{aligned}
&U(\nu)\sigma_0^x U(\nu)^{-1} \\
&= \sigma_0^x \exp \pi i \left[\frac{1+\sigma_1^z}{2}\frac{1-\sigma_2^z}{2}+\frac{1+\sigma_3^z}{2}\frac{1-\sigma_2^z}{2}-\frac{1-\sigma_4^z}{2}\frac{1-\sigma_3^z}{2}+\frac{1-\sigma_5^z}{2}\frac{1+\sigma_4^z}{2}+\frac{1-\sigma_5^z}{2}\frac{1+\sigma_6^z}{2}-\frac{1-\sigma_6^z}{2}\frac{1-\sigma_1^z}{2}\right] \\
&= \sigma_0^x \exp \frac{\pi i}{2}\left[\frac{1-\sigma_1^z\sigma_2^z}{2}+\frac{1-\sigma_2^z\sigma_3^z}{2}-\frac{1+\sigma_3^z\sigma_4^z}{2}+\frac{1-\sigma_4^z\sigma_5^z}{2}+\frac{1-\sigma_5^z\sigma_6^z}{2}-\frac{1+\sigma_6^z\sigma_1^z}{2}+\sigma_1^z-\sigma_2^z+\sigma_3^z-\sigma_4^z+\sigma_5^z-\sigma_6^z\right],
\end{aligned}
$$

but can this difference be realized by a 3-coboundary?

3 Computing $U(\nu)\sigma_0^x U(\nu)^{-1}$ for a given 3-cocycle

Solving the cocycle condition directly in Mathematica gives

$$
Z^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2 \times U(1) \times U(1). \tag{15}
$$

Putting $\omega(101) = x, \omega(110) = y$, we have

$$\omega(000) = 1, \tag{16}$$
$$\omega(001) = xy, \tag{17}$$
$$\omega(010) = 1, \tag{18}$$
$$\omega(011) = x^{-1}y^{-1}, \tag{19}$$
$$\omega(100) = y^{-1}, \tag{20}$$
$$\omega(111) = \pm x^{-1}. \tag{21}$$

Since we are interested in the nontrivial phase here, take $\omega(111) = -x^{-1}$. The homogeneous cocycle is given by

$$\nu(0, a, b, c) = \omega(a, -a+b, -b+c), \tag{22}$$

and therefore

$$\nu(0000) = 1, \tag{23}$$
$$\nu(0001) = xy, \tag{24}$$
$$\nu(0010) = x^{-1}y^{-1}, \tag{25}$$
$$\nu(0011) = 1, \tag{26}$$
$$\nu(0100) = y, \tag{27}$$
$$\nu(0101) = -x^{-1}, \tag{28}$$
$$\nu(0110) = x, \tag{29}$$
$$\nu(0111) = y^{-1}. \tag{30}$$

It follows that

$$\sum_{a_0a_1a_2} \nu(0a_0a_1a_2) \left|a_0a_1a_2\right\rangle \left\langle a_0a_1a_2\right| = \left|000\right\rangle \left\langle 000\right| + xy \left|001\right\rangle \left\langle 001\right| + x^{-1}y^{-1} \left|010\right\rangle \left\langle 010\right| + \left|011\right\rangle \left\langle 011\right| \tag{31}$$

$$+ y \left|100\right\rangle \left\langle 100\right| - x^{-1} \left|101\right\rangle \left\langle 101\right| + x \left|110\right\rangle \left\langle 110\right| + y^{-1} \left|111\right\rangle \left\langle 111\right|. \tag{32}$$

If we force this into an expression in spin operators, it becomes

$$= \frac{1+\sigma_0^z}{2}\frac{1+\sigma_1^z}{2}\frac{1+\sigma_2^z}{2} + xy\frac{1+\sigma_0^z}{2}\frac{1+\sigma_1^z}{2}\frac{1-\sigma_2^z}{2} + x^{-1}y^{-1}\frac{1+\sigma_0^z}{2}\frac{1-\sigma_1^z}{2}\frac{1+\sigma_2^z}{2} + \frac{1+\sigma_0^z}{2}\frac{1-\sigma_1^z}{2}\frac{1-\sigma_2^z}{2} \tag{33}$$

$$+ y\frac{1-\sigma_0^z}{2}\frac{1+\sigma_1^z}{2}\frac{1+\sigma_2^z}{2} - x^{-1}\frac{1-\sigma_0^z}{2}\frac{1+\sigma_1^z}{2}\frac{1-\sigma_2^z}{2} + x\frac{1-\sigma_0^z}{2}\frac{1-\sigma_1^z}{2}\frac{1+\sigma_2^z}{2} + y^{-1}\frac{1-\sigma_0^z}{2}\frac{1-\sigma_1^z}{2}\frac{1-\sigma_2^z}{2} \tag{34}$$

$$:= u(\sigma_0, \sigma_1, \sigma_2). \tag{35}$$

When the branching structure is chosen as in Fig. [a], the part of the unitary transformation that contributes to the transformation of σ_0^x is

$$\sum_{a_0,a_1,a_2,a_3,a_4,a_5,a_6} \nu(0a_0a_1a_2)\nu(0a_0a_3a_2)^{-1}\nu(0a_4a_0a_3)\nu(0a_5a_4a_0)^{-1}\nu(0a_5a_6a_0)\nu(0a_6a_0a_1)^{-1} \left|a_0a_1a_2a_3a_4a_5a_6\right\rangle \left\langle a_0a_1a_2a_3a_4 \right. \tag{36}$$

$$= u(\sigma_0, \sigma_1, \sigma_2)u(\sigma_0, \sigma_3, \sigma_2)^\dagger u(\sigma_4, \sigma_0, \sigma_3)u(\sigma_5, \sigma_4, \sigma_0)^\dagger u(\sigma_5, \sigma_6, \sigma_0)u(\sigma_6, \sigma_0, \sigma_1)^\dagger := \mathcal{U}. \tag{37}$$

On the other hand, the Levin–Gu model gives

$$B_0 = -\sigma_0^x \prod_{j=1}^{6} i^{\frac{1-\sigma_j^z\sigma_{j+1}^z}{2}} = i\sigma_0^x \prod_{j=1}^{6} \frac{1-i\sigma_j^z\sigma_{j+1}^z}{\sqrt{2}}. \tag{38}$$

Now solve

$$\mathcal{U}\sigma_0^x\mathcal{U}^\dagger = B_0. \tag{39}$$

No solution exists.

If the branching structure is chosen as in Fig. [b], then

$$\mathcal{U}' = u(\sigma_0,\sigma_1,\sigma_2)u(\sigma_0,\sigma_2,\sigma_3)u(\sigma_0,\sigma_3,\sigma_4)u(\sigma_0,\sigma_4,\sigma_5)u(\sigma_0,\sigma_5,\sigma_6)u(\sigma_0,\sigma_6,\sigma_1), \tag{40}$$

and solving

$$\mathcal{U}'\sigma_0^x\mathcal{U}'^\dagger = B_0 \tag{41}$$

again gives no solution. The choice in Fig. [b] breaks translation symmetry.

4 Appendix B in [2]

$$\tilde{Z} = \frac{1}{N_v}\sum_{g_0,g_1,g_2,g_3,\dots}\prod_{\text{link}}\tilde{d}(g_0,g_1)\prod_{\text{tetrahedron}}\tilde{G}(g_0,g_1,g_2,g_3), \tag{42}$$

where $\tilde{G}$ satisfies the $\mathbb{Z}_2$ symmetry

$$G(gg_0,gg_1,gg_2,gg_3) = G(g_0,g_1,g_2,g_3) \tag{43}$$

and is defined as follows:

$$\tilde{G}(\uparrow,\uparrow,\uparrow,\uparrow) = \tilde{G}(\downarrow,\downarrow,\downarrow,\downarrow) = G^{000}_{000} = 1, \tag{44}$$
$$\tilde{G}(\uparrow,\uparrow,\uparrow,\downarrow) = \tilde{G}(\downarrow,\downarrow,\downarrow,\uparrow) = G^{000}_{111} = -i, \tag{45}$$
$$\tilde{G}(\uparrow,\downarrow,\uparrow,\uparrow) = \tilde{G}(\downarrow,\uparrow,\downarrow,\downarrow) = G^{110}_{001} = -i, \tag{46}$$
$$\tilde{G}(\uparrow,\uparrow,\downarrow,\uparrow) = \tilde{G}(\downarrow,\downarrow,\uparrow,\downarrow) = G^{011}_{100} = -i, \tag{47}$$
$$\tilde{G}(\downarrow,\uparrow,\uparrow,\uparrow) = \tilde{G}(\uparrow,\downarrow,\downarrow,\downarrow) = G^{101}_{010} = -i, \tag{48}$$
$$\tilde{G}(\uparrow,\downarrow,\uparrow,\downarrow) = \tilde{G}(\downarrow,\uparrow,\downarrow,\uparrow) = G^{110}_{110} = -1, \tag{49}$$
$$\tilde{G}(\uparrow,\downarrow,\downarrow,\uparrow) = \tilde{G}(\downarrow,\uparrow,\uparrow,\downarrow) = G^{101}_{101} = -1, \tag{50}$$
$$\tilde{G}(\uparrow,\uparrow,\downarrow,\downarrow) = \tilde{G}(\downarrow,\downarrow,\uparrow,\uparrow) = G^{011}_{011} = -1. \tag{51}$$

Computing the cocycle condition shows that $\tilde{G}$ itself does not satisfy the cocycle condition. The function $\tilde{d}$ satisfies the $\mathbb{Z}_2$ symmetry $\tilde{d}(gg_0,gg_1) = \tilde{d}(g_0,g_1)$ and is defined by

$$\tilde{d}(\uparrow,\uparrow) = \tilde{d}(\downarrow,\downarrow) = d_0 = 1, \tag{52}$$
$$\tilde{d}(\uparrow,\downarrow) = \tilde{d}(\downarrow,\uparrow) = d_1 = -1. \tag{53}$$

The function $\tilde{d}$ does not appear in the model constructed from a cocycle $\nu \in Z^3(\mathbb{Z}_2, U(1))$ [1]. Whether the model obtained from $\tilde{G}$ and $\tilde{d}$ agrees with the Levin–Gu model is unclear; [2] only says, "*We expect that these two phases correspond to the two types of paramagnets H_0, H_1.*"

References

[1] Xie Chen, Zheng-Cheng Gu, Zheng-Xin Liu, Xiao-Gang Wen, *Symmetry protected topological orders and the group cohomology of their symmetry group*, arXiv:1106.4772.

[2] Michael Levin, Zheng-Cheng Gu, *Braiding statistics approach to symmetry-protected topological phases*, arXiv:1202.3120.

[3] Luiz H. Santos, Eduardo Fradkin, *Instanton effects in lattice models of bosonic symmetry-protected topological states*, arXiv:1512.06864.