Representing a Ground State of a Hamiltonian by Constraints from Local Terms

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Abstract

A memo on Proposition D.1 of Ref. [1].

A Hamiltonian

$$H = \sum_j H_j \tag{1}$$

is local if

$$||[H_j, \mathcal{O}_i]|| \leq u(|j-i|) \tag{2}$$

holds, where u is a function that decays faster than any power. Exponential decay is not required; we assume only faster-than-polynomial decay.

Let $H = \sum_j H_j$ be a gapped local Hamiltonian. Let the energy eigenvalue of the ground state $|\Psi\rangle$ be zero, and let $\Delta > 0$ be the gap. Then $|\Psi\rangle$ can be represented as a zero-energy eigenstate of certain local terms $\tilde{H}_j$, namely $\tilde{H}_j |\Psi\rangle = 0$.

We prove this. Define

$$\tilde{H}_j = \int_{-\infty}^{\infty} e^{iHt} H_j e^{-iHt} w(t)\, dt. \tag{3}$$

Then

$$\sum_j \tilde{H}_j = H \times \int_{-\infty}^{\infty} w(t)\, dt. \tag{4}$$

Thus we choose $w(t)$ so that

$$\int_{-\infty}^{\infty} w(t)\, dt = 1. \tag{5}$$

Equivalently, for the Fourier transform

$$\hat{w}(\epsilon) = \int_{-\infty}^{\infty} dt\, w(t) e^{i\epsilon t}, \tag{6}$$

we require

$$\hat{w}(0) = 1. \tag{7}$$

If we require $\tilde{H}_j$ to be Hermitian, then

$$w(t)^* = w(-t) \Leftrightarrow \hat{w}(\epsilon)^* = \hat{w}(-\epsilon). \tag{8}$$

For any excited state $|n\rangle$,

$$\left\langle n \,\middle|\, \tilde{H}_j \,\middle|\, \Psi \right\rangle = \int_{-\infty}^{\infty} e^{iE_n t} \langle n|H_j|\Psi\rangle \, w(t) \, dt = \hat{w}(E_n) \langle n|H_j|\Psi\rangle . \tag{9}$$

Therefore, if $\hat{w}(\epsilon)$ is chosen so that

$$\hat{w}(|\epsilon| > \Delta) = 0, \tag{10}$$

then, with P the projection onto the ground-state subspace, we obtain the desired property

$$(1-P)\tilde{H}_j P = 0. \tag{11}$$

Take $\hat{w}(\epsilon)$ to be a smooth C^∞ function satisfying

$$\hat{w}(\epsilon = 0) = 1, \quad \hat{w}(|\epsilon| > \Delta) = 0. \tag{12}$$

Then $w(t)$ decays faster than any power.

It remains to show that $\tilde{H}_j$ can be constructed to be local. For any local operator O_i supported near the site i,

$$||[\tilde{H}_j, O_i]|| \le \int_{-\infty}^{\infty} ||[H_j(t), O_i]|| \, |w(t)| \, dt. \tag{13}$$

By a Lieb–Robinson bound, roughly,

$$||[H_j(t), O_i]|| \le c||H_j|| ||O_i|| \times \max\{e^{-a(l-v|t|)}, 1\}, \quad l \sim |i-j|. \tag{14}$$

Hence we can estimate

$$||[\tilde{H}_j, O_i]|| \le c||H_j|| ||O_i|| \times 2 \times \left[\int_0^{l/v} e^{-a(l-vt)} |w(t)| \, dt + \int_{l/v}^{\infty} |w(t)| \, dt \right]. \tag{15}$$

We want to show that this decays faster than any power of l.

The following proof is due to Kenji Shimomura.[1] Since $w(t)$ decays faster than any power, for any $n \in \mathbb{Z}_{\ge 0}$,

$$|w(t)| t^n \xrightarrow{t \to \infty} 0. \tag{16}$$

Since $w(t)$ is bounded, for any $n \in \mathbb{Z}_{\ge 0}$ there exists a constant M_n such that

$$|w(t)| \le M_n t^{-n} \quad \text{for } t > 0. \tag{17}$$

Then

$$\int_{l/v}^{\infty} |w(t)| \, dt \le \int_{l/v}^{\infty} M_{n+1} t^{-n-1} \, dt = O(l^{-n}), \tag{18}$$

and

$$\begin{aligned}
\int_0^{l/v} e^{-a(l-vt)} |w(t)| \, dt &\le \int_0^{l/(2v)} e^{-a(l-vt)} M_0 \, dt + \int_{l/(2v)}^{l/v} e^{-a(l-vt)} M_{n+1} t^{-n-1} \, dt &(19)\\
&= M_0 \frac{e^{-al/2} - e^{-al}}{av} + M_{n+1} (l/v)^{-n} \int_{1/2}^{1} e^{-a(l-ls)} s^{-n-1} \, ds &(20)\\
&\le M_0 \frac{e^{-al/2} - e^{-al}}{av} + M_{n+1} (l/v)^{-n} \int_{1/2}^{1} s^{-n-1} \, ds &(21)\\
&= O(l^{-n}). &(22)
\end{aligned}$$

This proves the claim.

Memo:

- Lemma 2 of Ref. [2] also claims that $\tilde{H}_j$ is local in a more general situation where H is time dependent, but no proof is written there.

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[1] Kenji Shimomura, private communication.

References

[1] Alexei Kitaev, *Anyons in an exactly solved model and beyond*, math-ph/0507008.

[2] S. Bravyi and M. B. Hastings, *A short proof of stability of topological order under local perturbations*, arXiv:1001.4363.