

MPS and Symmetry (Draft)

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Abstract

This is a calculation memo for [1].

Consider the MPS

$$|\Psi_L\rangle = \sum_{n_1, \dots, n_L} \text{tr}[A_{n_1} \cdots A_{n_L}] |n_1 \cdots n_L\rangle. \quad (1)$$

Introduce

$$\mathcal{E}(X) := \sum_n A_n X A_n^\dagger, \quad \mathcal{E}^*(X) = \sum_n A_n^\dagger X A_n. \quad (2)$$

The matrices A_n can be chosen so that

$$\mathcal{E}(\mathbf{1}) = \mathbf{1}, \quad (3)$$

and

$$\mathcal{E}(\Lambda) = \Lambda. \quad (4)$$

Here Λ satisfies $\Lambda \geq 0$ and $\text{tr}(\Lambda) = 1$. The matrix Λ is Hermitian. **Check why one can choose the tensors in this form. Check whether positivity of Λ implies Hermiticity. Is Λ meant to be the set of all objects satisfying the above condition, or does it mean that such a Λ exists?**

- A necessary and sufficient condition for $|\Psi_L\rangle$ to be a pure state in an appropriate sense is that $\Lambda > 0$ and that \mathcal{E} has only one eigenvalue whose absolute value is 1.

Check this as well.

For a unitary transformation u acting on the local basis $|n\rangle$,

$$u = \sum_j e^{i\theta_j} |\tilde{j}\rangle \langle \tilde{j}|, \quad (5)$$

introduce

$$\begin{aligned} \mathcal{E}_u(X) &= \sum_{n, n'} \langle n' | u | n \rangle A_n X A_{n'}^\dagger \\ &= \sum_j e^{i\theta_j} \tilde{A}_j X \tilde{A}_j^\dagger, \end{aligned} \quad (6)$$

where

$$\tilde{A}_j = \sum_n \langle \tilde{j} | n \rangle A_n. \quad (7)$$

The inverse transformation is

$$A_n = \sum_j \langle n|\tilde{j}\rangle \tilde{A}_j. \quad (8)$$

Notice that

$$\begin{aligned} \sum_j \tilde{A}_j \tilde{A}_j^\dagger &= \sum_{j,n,n'} \langle \tilde{j}|n\rangle A_n \langle n'|\tilde{j}\rangle A_n^\dagger \\ &= \sum_n A_n A_n^\dagger = \mathbf{1}, \end{aligned} \quad (9)$$

$$\begin{aligned} \sum_j \tilde{A}_j^\dagger \Lambda \tilde{A}_j &= \sum_{j,n,n'} \langle n|\tilde{j}\rangle A_n^\dagger \Lambda \langle \tilde{j}|n'\rangle A_{n'} \\ &= \sum_n A_n^\dagger \Lambda A_n = \Lambda, \end{aligned} \quad (10)$$

$$\begin{aligned} \mathcal{E}(X) &= \sum_n A_n X A_n^\dagger \\ &= \sum_{n,j,j'} \langle n|\tilde{j}\rangle \tilde{A}_j X \langle \tilde{j}'|n\rangle \tilde{A}_j^\dagger = \sum_j \tilde{A}_j X \tilde{A}_j^\dagger, \end{aligned} \quad (11)$$

and

$$\begin{aligned} \mathcal{E}^*(X) &= \sum_n A_n^\dagger X A_n \\ &= \sum_{n,j,j'} \langle \tilde{j}|n\rangle \tilde{A}_j^\dagger X \langle n|\tilde{j}'\rangle \tilde{A}_j = \sum_j \tilde{A}_j^\dagger X \tilde{A}_j. \end{aligned} \quad (12)$$

Write the spectral radius of an operator A as

$$\rho(A) = \max_i |\lambda_i|. \quad (13)$$

Lemma. One has $\rho(\mathcal{E}_u) \leq 1$. Equality holds when there exist V and $e^{i\theta}$ such that

$$V^\dagger \tilde{A}_j = e^{i(\theta-\theta_j)} \tilde{A}_j V^\dagger \quad (14)$$

for all j . Moreover, \mathcal{E}_u has at most one eigenvalue of absolute value 1.

Proof— Let V be an eigenvector of \mathcal{E}_u with eigenvalue λ , i.e.

$$\sum_j e^{i\theta_j} \tilde{A}_j V \tilde{A}_j^\dagger = \lambda V. \quad (15)$$

Multiplying by ΛV^\dagger from the right gives

$$\lambda V \Lambda V^\dagger = \sum_j e^{i\theta_j} \tilde{A}_j V \tilde{A}_j^\dagger \Lambda V^\dagger. \quad (16)$$

Taking the trace and absolute value, and using positivity of $V^\dagger V$, we get

$$\begin{aligned} |\lambda| \text{tr}[V \Lambda V^\dagger] &= \left| \sum_j e^{i\theta_j} \text{tr}[\tilde{A}_j V \tilde{A}_j^\dagger \Lambda V^\dagger] \right| \\ &\leq \sum_j \left| \text{tr}[\tilde{A}_j V \tilde{A}_j^\dagger \Lambda V^\dagger] \right|. \end{aligned} \quad (17)$$

Equality here holds when the $U(1)$ phases of the complex numbers

$$e^{i\theta_j} \operatorname{tr} [\tilde{A}_j V \tilde{A}_j^\dagger \Lambda V^\dagger] \quad (18)$$

are aligned, namely when

$$\operatorname{Arg} \left(e^{i\theta_j} \operatorname{tr} [\tilde{A}_j V \tilde{A}_j^\dagger \Lambda V^\dagger] \right) = \theta_j + \operatorname{Arg} \left(\operatorname{tr} [\tilde{A}_j V \tilde{A}_j^\dagger \Lambda V^\dagger] \right) \equiv \theta. \quad (19)$$

Also note that

$$\begin{aligned} \operatorname{tr} [\tilde{A}_j V \tilde{A}_j^\dagger \Lambda V^\dagger] &= \operatorname{tr} [V \tilde{A}_j^\dagger \Lambda V^\dagger \tilde{A}_j] \\ &= \operatorname{tr} [V \tilde{A}_j^\dagger \Lambda^{\frac{1}{2}} \Lambda^{\frac{1}{2}} V^\dagger \tilde{A}_j] \\ &= \left\langle \Lambda^{\frac{1}{2}} \tilde{A}_j V^\dagger, \Lambda^{\frac{1}{2}} V^\dagger \tilde{A}_j \right\rangle, \end{aligned} \quad (20)$$

where the inner product is $\langle A, B \rangle = \operatorname{tr} [A^\dagger B]$. By the Cauchy–Schwarz inequality,

$$\begin{aligned} \left| \operatorname{tr} [\tilde{A}_j V \tilde{A}_j^\dagger \Lambda V^\dagger] \right| &= \left| \left\langle \Lambda^{\frac{1}{2}} \tilde{A}_j V^\dagger, \Lambda^{\frac{1}{2}} V^\dagger \tilde{A}_j \right\rangle \right| \\ &\leq \left| \left\langle \Lambda^{\frac{1}{2}} \tilde{A}_j V^\dagger, \Lambda^{\frac{1}{2}} \tilde{A}_j V^\dagger \right\rangle \right|^{\frac{1}{2}} \left| \left\langle \Lambda^{\frac{1}{2}} V^\dagger \tilde{A}_j, \Lambda^{\frac{1}{2}} V^\dagger \tilde{A}_j \right\rangle \right|^{\frac{1}{2}} \\ &= \left| \operatorname{tr} [V \tilde{A}_j^\dagger \Lambda \tilde{A}_j V^\dagger] \right|^{\frac{1}{2}} \left| \operatorname{tr} [\tilde{A}_j^\dagger V \Lambda V^\dagger \tilde{A}_j] \right|^{\frac{1}{2}}. \end{aligned} \quad (21)$$

Equality holds if one of the two vectors vanishes, or if

$$\Lambda^{\frac{1}{2}} \tilde{A}_j V^\dagger = \alpha_j \Lambda^{\frac{1}{2}} V^\dagger \tilde{A}_j, \quad \alpha_j \in \mathbb{C}. \quad (22)$$

In that case, taking norms of both sides gives

$$\left| \operatorname{tr} [V \tilde{A}_j^\dagger \Lambda \tilde{A}_j V^\dagger] \right| = |\alpha_j|^2 \left| \operatorname{tr} [\tilde{A}_j^\dagger V \Lambda V^\dagger \tilde{A}_j] \right|. \quad (23)$$

Also, in this case

$$\begin{aligned} \operatorname{tr} [\tilde{A}_j V \tilde{A}_j^\dagger \Lambda V^\dagger] &= \operatorname{tr} [V \tilde{A}_j^\dagger \Lambda V^\dagger \tilde{A}_j] \\ &= \operatorname{tr} [(\Lambda^{\frac{1}{2}} \tilde{A}_j V^\dagger)^\dagger \Lambda^{\frac{1}{2}} V^\dagger \tilde{A}_j] \\ &= \alpha_j^* \operatorname{tr} [(\Lambda^{\frac{1}{2}} V^\dagger \tilde{A}_j)^\dagger \Lambda^{\frac{1}{2}} V^\dagger \tilde{A}_j], \end{aligned} \quad (24)$$

and hence

$$\operatorname{Arg} \left(\operatorname{tr} [\tilde{A}_j V \tilde{A}_j^\dagger \Lambda V^\dagger] \right) = -\operatorname{Arg}(\alpha_j). \quad (25)$$

Applying the Cauchy–Schwarz inequality once more,

$$\begin{aligned} &\sum_j \left| \operatorname{tr} [V \tilde{A}_j^\dagger \Lambda \tilde{A}_j V^\dagger] \right|^{\frac{1}{2}} \left| \operatorname{tr} [\tilde{A}_j^\dagger V \Lambda V^\dagger \tilde{A}_j] \right|^{\frac{1}{2}} \\ &\leq \left(\sum_j \left| \operatorname{tr} [V \tilde{A}_j^\dagger \Lambda \tilde{A}_j V^\dagger] \right| \right)^{\frac{1}{2}} \left(\sum_j \left| \operatorname{tr} [\tilde{A}_j^\dagger V \Lambda V^\dagger \tilde{A}_j] \right| \right)^{\frac{1}{2}}. \end{aligned} \quad (26)$$

Equality holds when there exists a constant $r \in \mathbb{R}$, independent of j , such that

$$\left| \operatorname{tr} [V \tilde{A}_j^\dagger \Lambda \tilde{A}_j V^\dagger] \right|^{\frac{1}{2}} = r \left| \operatorname{tr} [\tilde{A}_j^\dagger V \Lambda V^\dagger \tilde{A}_j] \right|^{\frac{1}{2}}. \quad (27)$$

Now note that

$$\sum_j \operatorname{tr} [V \tilde{A}_j^\dagger \Lambda \tilde{A}_j V^\dagger] = \operatorname{tr} [V \Lambda V^\dagger], \quad (28)$$

and

$$\sum_j \text{tr} [\tilde{A}_j^\dagger V \Lambda V^\dagger \tilde{A}_j] = \sum_j \text{tr} [\tilde{A}_j \tilde{A}_j^\dagger V \Lambda V^\dagger] = \text{tr} [V \Lambda V^\dagger]. \quad (29)$$

Therefore

$$|\lambda| \text{tr} [V \Lambda V^\dagger] \leq \text{tr} [V \Lambda V^\dagger]. \quad (30)$$

Thus, unless $\text{tr} [V \Lambda V^\dagger] = 0$, one obtains $|\lambda| \leq 1$. This proves the first part, $\rho(\mathcal{E}_u) \leq 1$.

The equality condition $|\lambda| = 1$ is as follows. Write $\alpha_j = |\alpha_j| e^{i\phi_j}$. Then $|\alpha_j| = r$ is common to all j , and from

$$\theta_j - \text{Arg}(\alpha_j) \equiv \theta \quad (31)$$

we see that α_j is ultimately independent of j in the form

$$\alpha_j = r e^{-i(\theta - \theta_j)}. \quad (32)$$

Moreover, since $|\alpha_j|$ is independent of j , summing (27) over j gives $r = 1$. Hence the equality condition is

$$\Lambda^{\frac{1}{2}} \tilde{A}_j V^\dagger = e^{-i(\theta - \theta_j)} \Lambda^{\frac{1}{2}} V^\dagger \tilde{A}_j. \quad (33)$$

Since Λ is invertible, this gives the latter part of the assertion,

$$\tilde{A}_j V^\dagger = e^{-i(\theta - \theta_j)} V^\dagger \tilde{A}_j. \quad (34)$$

At this point,

$$\mathcal{E}_u(V) = \sum_j e^{i\theta} \tilde{A}_j \tilde{A}_j^\dagger V = e^{i\theta} V, \quad (35)$$

so the eigenvalue is $\lambda = e^{i\theta}$. Furthermore, V is unitary as follows:

$$\mathcal{E}(V^\dagger V) = \sum_j \tilde{A}_j V^\dagger V \tilde{A}_j^\dagger = \sum_j V^\dagger \tilde{A}_j \tilde{A}_j^\dagger V = V^\dagger V. \quad (36)$$

By the assumption of purity, $\mathbf{1}$ is the unique eigenvector of \mathcal{E} with eigenvalue of absolute value 1, so $V^\dagger V = \mathbf{1}$. \square

Lemma. Moreover, \mathcal{E}_u has at most one eigenvalue of absolute value 1.

Proof— We show uniqueness. Let V, V' be eigenvectors of \mathcal{E}_u with eigenvalues $e^{i\theta}$ and $e^{i\theta'}$, respectively. Then

$$\begin{aligned} \mathcal{E}(V^\dagger V') &= \sum_j \tilde{A}_j V^\dagger V' \tilde{A}_j^\dagger \\ &= \sum_j e^{i(\theta_j - \theta)} V^\dagger \tilde{A}_j e^{-i(\theta_j - \theta')} \tilde{A}_j^\dagger V' \\ &= e^{i(\theta' - \theta)} \sum_j V^\dagger \tilde{A}_j \tilde{A}_j^\dagger V' = e^{i(\theta' - \theta)} V^\dagger V'. \end{aligned} \quad (37)$$

Again by the purity assumption, the only eigenvector of \mathcal{E} with eigenvalue of absolute value 1 is $\mathbf{1}$. Hence

$$V^\dagger V' = \mathbf{1}, \quad (38)$$

and

$$e^{i(\theta' - \theta)} = 1. \quad (39)$$

□

Lemma. Furthermore,

$$V\Lambda = \Lambda V. \quad (40)$$

Proof— We have

$$\mathcal{E}^*(V\Lambda V^\dagger) = \sum_j \tilde{A}_j^\dagger V\Lambda V^\dagger \tilde{A}_j. \quad (41)$$

Using (14),

$$\mathcal{E}^*(V\Lambda V^\dagger) = \sum_j V\tilde{A}_j^\dagger \Lambda \tilde{A}_j V^\dagger = V\Lambda V^\dagger. \quad (42)$$

On the other hand, considering a Jordan decomposition and arguing as for \mathcal{E} , the eigenvector of \mathcal{E}^* with eigenvalue $\lambda = 1$ is also unique. Hence there exists $z \in \mathbb{C}$ such that

$$V\Lambda V^\dagger = z\Lambda. \quad (43)$$

Using the condition $\text{tr}[\Lambda] = 1$, we obtain the desired assertion. □

References

- [1] D. Perez-Garcia, M. M. Wolf, M. Sanz, F. Verstraete, and J. I. Cirac, *String Order and Symmetries in Quantum Spin Lattices*, Phys. Rev. Lett. **100**, 167202 (2008).