On the Behavior of Transfer Matrices between MPS

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1 Preliminaries

I will call a collection of $D \times D$ matrices $\{A_i\}_{i=1}^d$, $A_i \in \mathrm{Mat}_{D\times D}(\mathbb{C})$, an MPS. When matrix sizes D of different values are treated at the same time, I will write, for instance, a D-MPS. An MPS is called canonical if

$$\sum_i A_i A_i^\dagger = 1_D \tag{1}$$

is satisfied. An MPS is called injective if, denoting by ρ_A the spectral radius of the transfer matrix

$$T_{AA}(X) = \sum_i A_i X A_i^\dagger, \tag{2}$$

the eigenvalue λ with $|\lambda| = \rho_A$ is unique, the corresponding eigenspace is one-dimensional, and the eigenvector is positive definite. For an injective MPS $\{A_i\}_i$, note that it can always be brought into canonical form by using the eigenvector X, $X > 0$, for $\lambda = \rho_A$ and performing

$$A_i \mapsto \frac{1}{\sqrt{\rho_A}} X^{-1/2} A_i X^{1/2}. \tag{3}$$

The following fact is known [1]. Write

$$|\{A_i\}_i\rangle_L := \sum_{i_1,\dots,i_L} \mathrm{tr}\left[A_{i_1}\cdots A_{i_L}\right] |i_1\cdots i_L\rangle\,. \tag{4}$$

For two canonical and injective D-MPS $\{A_i\}_i$ and $\{B_i\}_i$, if there exist $L \in \mathbb{N}$ and $e^{i\phi} \in U(1)$ such that $|\{A_i\}_i\rangle_L = e^{i\phi}\,|\{B_i\}_i\rangle_L$, then there exist $e^{i\theta} \in U(1)$ and $V \in U(D)$ such that

$$A_i = e^{i\theta} V^\dagger B_i V. \tag{5}$$

The phase $e^{i\theta}$ is unique, and V is unique up to a $U(1)$ phase.

Let G be a finite group, and let $\phi : G \to \{\pm 1\}$ be a homomorphism. The value ϕ_g indicates whether the action is unitary or antiunitary. For a matrix X, introduce the notation

$$X^{\phi_g} = \left\{\begin{array}{ll} X & (\phi_g = 1), \\ X^* & (\phi_g = -1). \end{array}\right. \tag{6}$$

On the Hilbert space spanned by $\bigotimes_{x=1}^L |i_x\rangle$, define the G action by

$$\hat g = \bigotimes_{x=1}^L \hat u_g^{[x]}, \quad \hat u_g^{[x]}\,|j_x\rangle = \sum_{i_x} |i_x\rangle\,[u_g]_{i_x j_x}, \quad u_g \in U(d), \quad u_g u_h^{\phi_g} = u_{gh}. \tag{7}$$

The collection $\{u_g\}_{g\in G}$ is a linear representation of G. Define a canonical and injective MPS $\{A_i\}_i$ to be G-symmetric if, for every $L \in \mathbb{N}$, there exists $e^{i\phi_L}$ such that

$$\hat g\,|\{A_i\}_i\rangle_L = e^{i\phi_L}\,|\{A_i\}_i\rangle_L\,. \tag{8}$$

Then, by the fact stated above, for each $g \in G$ there exist $e^{i\theta_g} \in U(1)$ and $V_g \in U(D)$ such that

$$\sum_j [u_g]_{ij} A_j^{\phi_g} = e^{i\theta_g} V_g^\dagger A_i V_g. \tag{9}$$

Here $e^{i\theta_g}$ is unique, and V_g is unique up to a $U(1)$ phase. The relation

$$e^{i\theta_g} e^{i\phi_g \theta_h} = e^{i\theta_{gh}}, \quad g, h \in G \tag{10}$$

holds. Moreover, there exists $\omega_{g,h} \in U(1)$ such that

$$V_g V_h^{\phi_g} = \omega_{g,h} V_{gh}, \quad g, h \in G. \tag{11}$$

Then

$$(\omega_{h,k})^{\phi_g} \omega_{gh,k}^{-1} \omega_{g,hk} \omega_{g,h}^{-1} = 1, \tag{12}$$

that is, $\omega \in Z^2(G, U(1)_\phi)$. Under a redefinition of the phases of V_g, $V_g \mapsto V_g \alpha_g$, $\alpha_g \in U(1)$, one has $\omega \mapsto \omega\, \delta\alpha$. Thus ω is classified by the group cohomology class $[\omega] \in H^2(G, U(1)_\phi)$.

2 Transfer matrices

For a canonical and injective D_0-MPS $\{A_i^0\}_{i=1}^d$ and a D_1-MPS $\{A_i^1\}_i$, define the transfer matrix $T_{A^0 A^1} \in \mathrm{End}(\mathrm{Mat}_{D_0 \times D_1}(\mathbb{C}))$ by

$$T_{A^0 A^1}(X) = \sum_i A_i^0 X (A_i^1)^\dagger. \tag{13}$$

The inner product between the MPS on L sites is $\mathrm{tr}\,[(T_{A^0 A^1})^L]$. If we choose the basis $e_{ab} = |a\rangle \langle b|$ for the space of $D_0 \times D_1$ matrices, the matrix elements of the transfer matrix are

$$[T_{A^0 A^1}]_{ab,cd} = \mathrm{tr}\,[\sum_i A_i^0 e_{cd} (A_i^1)^\dagger e_{ab}^T] = \sum_i \langle a|A_i^0|c\rangle \langle b|(A_i^1)^*|d\rangle\,. \tag{14}$$

When $\{A_i^0\}_i$ and $\{A_i^1\}_i$ are G-symmetric, let us examine the G symmetry of the transfer matrix $T_{A^0 A^1}$. Assume

$$\sum_j [u_g]_{ij} (A_j^\mu)^{\phi_g} = e^{i\theta_g^\mu} (V_g^\mu)^\dagger A_i^\mu V_g^\mu, \quad V_g^\mu (V_h^\mu)^{\phi_g} = \omega_{g,h}^\mu V_{gh}^\mu, \quad \mu \in \{0, 1\}. \tag{15}$$

Note that

$$A_i^\mu = \left(\sum_j [u_g^\dagger]_{ij} e^{i\theta_g^\mu} (V_g^\mu)^\dagger A_j^\mu V_g^\mu \right)^{\phi_g}. \tag{16}$$

Then, for $g \in G$,

$$\begin{aligned}
T_{A^0 A^1}(X) &= \sum_i A_i^0 X (A_i^1)^\dagger &(17)\\
&= \sum_i \left(\sum_j [u_g^\dagger]_{ij} e^{i\theta_g^0} (V_g^0)^\dagger A_j^0 V_g^0 \right)^{\phi_g} X \left(\left(\sum_k [u_g^\dagger]_{ik} e^{i\theta_g^1} (V_g^1)^\dagger A_k^1 V_g^1 \right)^{\phi_g} \right)^\dagger &(18)\\
&= \sum_i \left(e^{i\theta_g^0} (V_g^0)^\dagger A_i^0 V_g^0 \right)^{\phi_g} X \left(\left(e^{i\theta_g^1} (V_g^1)^\dagger A_i^1 V_g^1 \right)^{\phi_g} \right)^\dagger &(19)\\
&= e^{i\phi_g(\theta_g^0 - \theta_g^1)} ((V_g^0)^\dagger)^{\phi_g} \left(\sum_i A_i^0 V_g^0 X^{\phi_g} (V_g^1)^\dagger (A_i^1)^\dagger \right)^{\phi_g} (V_g^1)^{\phi_g} &(20)\\
&= e^{i\phi_g(\theta_g^0 - \theta_g^1)} ((V_g^0)^\dagger)^{\phi_g} \left(T_{A^0 A^1}(V_g^0 X^{\phi_g} (V_g^1)^\dagger) \right)^{\phi_g} (V_g^1)^{\phi_g}. &(21)
\end{aligned}$$

In other words, the following symmetry holds:

$$((V_g^0))^{\phi_g} T_{A^0A^1}(X)((V_g^1)^{\phi_g})^\dagger = e^{i\phi_g(\theta_g^0-\theta_g^1)} \left(T_{A^0A^1}(V_g^0 X^{\phi_g}(V_g^1)^\dagger)\right)^{\phi_g}. \tag{22}$$

In components,

$$\begin{aligned}
(\mathrm{lhs})_{ab,cd} &= \mathrm{Tr}[((V_g^0))^{\phi_g} T_{A^0A^1}(e_{cd})((V_g^1)^{\phi_g})^\dagger e_{ab}^T] && (23)\\
&= \sum_{ef} [T_{A^0A^1}]_{ef,cd} \mathrm{Tr}[((V_g^0))^{\phi_g} e_{ef}((V_g^1)^{\phi_g})^\dagger e_{ab}^T] && (24)\\
&= \sum_{ef} [T_{A^0A^1}]_{ef,cd} [V_g^0]_{ae}^{\phi_g} [(V_g^1)^*]_{bf}^{\phi_g}, && (25)
\end{aligned}$$

while

$$\begin{aligned}
(\mathrm{rhs})_{ab,cd} &= \mathrm{Tr}\left[e^{i\phi_g(\theta_g^0-\theta_g^1)} \left(T_{A^0A^1}(V_g^0 e_{cd}^{\phi_g}(V_g^1)^\dagger)\right)^{\phi_g} e_{ab}^T\right] && (26)\\
&= e^{i\phi_g(\theta_g^0-\theta_g^1)} \sum_{ef} \mathrm{Tr}\left[\left(T_{A^0A^1}(e_{ee}V_g^0 e_{cd}^{\phi_g}(V_g^1)^\dagger e_{ff})\right)^{\phi_g} e_{ab}^T\right] && (27)\\
&= e^{i\phi_g(\theta_g^0-\theta_g^1)} \sum_{ef} [V_g^0]_{ec}^{\phi_g} [(V_g^1)^*]_{fd}^{\phi_g} \mathrm{Tr}[(T_{A^0A^1}(e_{ef}))^{\phi_g} e_{ab}^T] && (28)\\
&= e^{i\phi_g(\theta_g^0-\theta_g^1)} \sum_{ef} [V_g^0]_{ec}^{\phi_g} [(V_g^1)^*]_{fd}^{\phi_g} [T_{A^0A^1}]_{ab,ef}^{\phi_g}. && (29)
\end{aligned}$$

Thus

$$\sum_{ef} [T_{A^0A^1}]_{ef,cd}^{\phi_g} [V_g^0]_{ae} [(V_g^1)^*]_{bf} = e^{i(\theta_g^0-\theta_g^1)} \sum_{ef} [V_g^0]_{ec} [(V_g^1)^*]_{fd} [T_{A^0A^1}]_{ab,ef}, \tag{30}$$

and, written as

$$\sum_{ef} (V_g^0 \otimes V_g^{1*})_{ab,ef} [T_{A^0A^1}]_{ef,cd}^{\phi_g} = e^{i(\theta_g^0-\theta_g^1)} \sum_{ef} [T_{A^0A^1}]_{ab,ef} (V_g^0 \otimes V_g^{1*})_{ef,cd}, \tag{31}$$

we see that this has the following matrix form:

$$(V_g^0 \otimes V_g^{1*})[T_{A^0A^1}]^{\phi_g} = e^{i(\theta_g^0-\theta_g^1)} T_{A^0A^1}(V_g^0 \otimes V_g^{1*}), \quad g \in G. \tag{32}$$

Let $|\lambda\rangle$ be an eigenstate of $T_{A^0A^1}$ with eigenvalue λ.

If $e^{i\theta_g^0}$ and $e^{i\theta_g^1}$ are different one-dimensional representations of G, then there exists $g \in G$ with $\phi_g = 1$ such that $e^{i(\theta_g^0-\theta_g^1)} \neq 1$. For such a $g \in G$, $e^{-i(\theta_g^0-\theta_g^1)}\lambda$ is also an eigenvalue.

Suppose that $e^{i\theta_g^0}$ and $e^{i\theta_g^1}$ are the same one-dimensional representation of G. Then the transfer matrix $T_{A^0A^1}$ satisfies the symmetry

$$(V_g^0 \otimes V_g^{1*})[T_{A^0A^1}]^{\phi_g} = T_{A^0A^1}(V_g^0 \otimes V_g^{1*}), \quad g \in G. \tag{33}$$

If $[\omega^0] \neq [\omega^1]$, then $T_{A^0A^1}$ is block diagonalized with respect to the irreducible decomposition of $V^0 \otimes V^{1*}$. Since the dimensions of the $\omega^0\omega^{1*}$-irreducible representations are at least two, if λ is an eigenvalue, then either λ itself, or λ^*, is also an eigenvalue.

From the above, if $e^{i\theta_g^0}$ and $e^{i\theta_g^1}$ are different one-dimensional representations, or if $[\omega^0] \neq [\omega^1]$, then the absolute values of the eigenvalues of the transfer matrix $T_{A^0A^1}$ always appear in pairs of at least two.

In other words, when two translation-invariant and G-symmetric one-dimensional nondegenerate states $\{A_i^0\}_i$ and $\{A_i^1\}_i$ belong to different SPT phases, the absolute values of the eigenvalues of the transfer matrix $T_{A^0A^1}$ always appear in pairs of at least two.

References

[1] D. Perez-Garcia, F. Verstraete, M. M. Wolf, J. I. Cirac, *Matrix Product State Representations*, arXiv:quant-ph/0608197.