Derivation of the 54 and 38 Symmetry Classes for Non-Hermitian Hamiltonians*

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1 The 54 symmetry classes

Let G be a finite group. Let $\phi, \eta, c : G \to \mathbb{Z}_2 = \{\pm 1\}$ be homomorphisms. For a square matrix H, consider symmetries of the following form:

$$u_g \left\{ \begin{array}{ll} H & (\phi_g, \eta_g) = (1,1) \\ H^* & (\phi_g, \eta_g) = (-1,1) \\ H^\dagger & (\phi_g, \eta_g) = (1,-1) \\ H^T & (\phi_g, \eta_g) = (-1,-1) \end{array} \right\} u_g^\dagger = c_g H, \quad u_g \left\{ \begin{array}{ll} u_h & (\phi_g = 1) \\ u_h^* & (\phi_g = -1) \end{array} \right\} = z_{g,h} u_{gh}. \tag{1}$$

Here u_g is a unitary matrix, and $z_{g,h}$ is a 2-cocycle $z \in Z^2(G, U(1)_\phi)$. The matrix H is block diagonalized into sectors of irreducible representations of the subgroup $G_0 = \{g \in G | \phi_g = \eta_g = c_g = 1\}$. For a group element $g \in G \backslash G_0$, the general theory says that either it changes the irreducible representation, or it acts within an irreducible representation as a $\mathbb{Z}_2$ symmetry. In the latter case, when $\phi_g = -1$, there are two possibilities classified by $u_g u_g^* = \pm 1$.

We are interested in how many inequivalent symmetries exist among those that do not change the irreducible representation. We may assume $G = \mathbb{Z}_2^{\times N}$. We may also assume that there is no group element with $(\phi_g, c_g, \eta_g) = (1,1,1)$, so it is enough to consider only $N = 0, 1, 2, 3$.

Let us record the correspondence between (ϕ_g, c_g, η_g) and the names of the symmetries in [1].

(ϕ_g, c_g, η_g)		
$(-1,1,1)$	$uH^*u^\dagger = H$	TRS
$(-1,1,-1)$	$uH^Tu^\dagger = H$	TRS†
$(-1,-1,-1)$	$uH^Tu^\dagger = -H$	PHS
$(-1,-1,1)$	$uH^*u^\dagger = -H$	PHS†
$(1,-1,1)$	$uHu^\dagger = -H$	SLS (sublattice symmetry)
$(1,1,-1)$	$uH^\dagger u^\dagger = H$	pH (pseudo-Hermiticity)
$(1,-1,-1)$	$uH^\dagger u^\dagger = -H$	CS (chiral symmetry)

— The case $N = 0$.
There is one possibility.

— The case $N = 1$.
The following seven possibilities exhaust the cases:

(ϕ_g, c_g, η_g)	
$(-1,1,1)$	TRS
$(-1,1,-1)$	TRS†
$(-1,-1,-1)$	PHS
$(-1,-1,1)$	PHS†
$(1,-1,1)$	SLS
$(1,1,-1)$	pH
$(1,-1,-1)$	CS

*Mistakes corrected on August 22, 2025.

For $\phi_g = -1$, there are two possibilities, $u_1 u_1^* = \pm 1$. Thus there are $2 \times 4 + 3 = 11$ possibilities.

— The case $N = 2$.
When an element with $\phi_1 = -1$ is included, we may fix the generator to have $\phi_g = -1$, namely to be TRS, PHS, TRS†, or PHS†. The following six possibilities exhaust the cases:

(ϕ_g, c_g, η_g)	symmetry	equivalent symmetry
$(-1,1,1),(-1,-1,-1)$	TRS+PHS	
$(-1,1,-1),(-1,-1,1)$	TRS†+PHS†	
$(-1,1,1),(-1,1,-1)$	TRS+TRS†	TRS+pH
$(-1,1,1),(-1,-1,1)$	TRS+PHS†	TRS+SLS
$(-1,-1,-1),(-1,-1,1)$	PHS+TRS†	PHS+pH
$(-1,-1,-1),(-1,1,-1)$	PHS+PHS†	PHS+SLS

The relative phase can be fixed so that $u_1 u_2^* = u_2 u_1^*$. Thus, for each of the six possibilities, there are 2^2 choices of signs $u_1 u_1^* = \pm 1, u_2 u_2^* = \pm 1$. If no element with $\phi_g = -1$ is included, there is only the following one possibility:

(ϕ_g, c_g, η_g)	symmetry	equivalent symmetry
$(1,-1,1),(1,1,-1)$	SLS+pH	CS+SLS, CS+pH

In this case there are two choices for the sign in the commutation relation $u_1 u_2 = \pm u_2 u_1$. Hence there are $2^2 \times 6 + 2 = 26$ possibilities.

— The case $N = 3$.
We may choose the generators so that one of them has $\phi_g = -1$. There is only the following one possibility:

(ϕ_g, c_g, η_g)	symmetry	equivalent symmetry
$(-1,1,1),(-1,-1,1),(-1,1,-1)$	TRS+PHS†+TRS†	TRS+PHS+SLS, TRS+PHS+pH

Taking the generators to be TRS (T), PHS (C), and SLS (S), we can fix the relative phase of u_T, u_C by $u_T u_C^* = u_C u_T^*$. We can also fix $u_S^2 = 1$. There are choices of signs $u_T u_T^*, u_C u_C^* = \pm 1$ and of signs in the commutation relations $u_T u_S^* = \pm u_S u_T, u_C u_S^* = \pm u_S u_C$, so there are $2^4 = 16$ possibilities.

Therefore we obtain $1 + 11 + 26 + 16 = 54$ symmetry classes.

Furthermore, by the equivalence of symmetry classes induced by $H \mapsto iH$, one obtains the independent 38 symmetry classes of [1]. The calculation is omitted here.

If one is interested only in the classification for point gaps, it is enough to consider the 38 symmetry classes. When real-line gaps and imaginary-line gaps are considered simultaneously, or when one wants to fix the symmetry classes of the corresponding Hermitian and anti-Hermitian systems, the above 54 symmetry classes are convenient.

Looking at the classification above, the 54 symmetry classes can be specified by AZ classes, AZ† classes, and AZ+SLS/pH classes [1].

2 Counting the number of independent symmetry classes

Here we discuss a method for calculating the number of independent symmetry classes, including the case with at most p $\mathbb{Z}_2$ unitary symmetries.

Possible homomorphisms

$$(\phi, \eta, c) : \mathbb{Z}_2^n \mapsto \mathbb{Z}_2^3 \tag{2}$$

are specified by an $n \times 3$ matrix

$$M = \begin{pmatrix} \phi_1 & \eta_1 & c_1 \\ \vdots & & \\ \phi_n & \eta_n & c_n \end{pmatrix} \in \mathrm{Mat}_{n\times 3}(\mathbb{Z}_2), \tag{3}$$

whose rows are the $\mathbb{Z}_2 = \{0,1\}$-valued row vectors (ϕ_j, η_j, c_j) for the n generators $1,\dots,n$. Since $(0,0,0)$ is a $\mathbb{Z}_2$ unitary symmetry, the condition that the number of $\mathbb{Z}_2$ unitary symmetries is at most p is

$$\mathrm{rank}_{\mathbb{Z}_2} M \geq n-p \tag{4}$$

over the field $\mathbb{Z}_2$. Since $\mathrm{rank} M \leq 3$, it is enough to consider $n \leq p+3$. A change of basis of $\mathbb{Z}_2^n$ corresponds to elementary row operations over $\mathbb{Z}_2$ on M:

$$M \mapsto VM, \quad V \in GL_n(\mathbb{Z}_2). \tag{5}$$

Matrices related by elementary row operations are identified, so by putting M into row-reduced standard form, one obtains a unique representative of each equivalence class. We denote the standard form to which M belongs by

$$c(M). \tag{6}$$

The number of independent factor systems is given by $|H^2(\mathbb{Z}_2^n, U(1)_\phi)|$.

- When $\phi_g = 0$ for all $g \in \mathbb{Z}_2^n$: The factor system is characterized by the commutation relations for the generators $(1,\dots,n)$,

  $$u_i u_j = (-1)^{\epsilon_{i,j}} u_j u_i, \quad \epsilon_{i,j} \in \mathbb{Z}_2. \tag{7}$$

  There are as many independent $\mathbb{Z}_2$ choices as the number of pairs, $n(n-1)/2$, so the number is

  $$2^{n(n-1)/2}. \tag{8}$$

  We take ϵ to be symmetric with zero diagonal entries:

  $$\epsilon^\top = \epsilon, \quad \epsilon_{i,i} = 0, \quad i = 1,\dots,n. \tag{9}$$

- When there exists $g \in \mathbb{Z}_2^n$ such that $\phi_g = 1$: We may assume $\phi_1 = 1, \phi_{i>1} = 0$. Moreover, we may fix

  $$u_i^2 = 1, \quad i = 2,\dots,n. \tag{10}$$

  Then, in addition to the commutation relations for the generators $(1,\dots,n)$,

  $$u_i (u_j)^{\phi_i} = (-1)^{\epsilon_{i,j}} u_j u_i, \quad \epsilon_{i,j} \in \mathbb{Z}_2, \tag{11}$$

  [1] there is one independent invariant $u_1 u_1^* = (-1)^{\theta_1}$. Thus the number is

  $$2^{n(n-1)/2+1}. \tag{12}$$

This gives a way to compute, for any $p \in \mathbb{Z}_{\geq 0}$, the number of independent $\mathbb{Z}_2^N$-type symmetry classes of non-Hermitian systems containing at most p $\mathbb{Z}_2$ unitary symmetries.

2.1 Identification by $H \mapsto iH$

The transformation $H \mapsto iH$ induces

$$\iota : (\phi_g, \eta_g, c_g) \mapsto (\phi_g, \eta_g, c_g + \phi_g + \eta_g). \tag{13}$$

It induces the following transformation on the matrix M:

$$M \mapsto \iota M = M \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{14}$$

There are the following two possibilities.

[1] We have introduced the notation

$$m^{\phi_g} = \begin{cases} m & (\phi_g = 0) \\ m^* & (\phi_g = 1) \end{cases}.$$

2.1.1 The case where ιM is inequivalent to M

This is the case where ιM and M are not related by a change of basis, namely $c(\iota M) \neq c(M)$.

For example,

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \tag{15}$$

In this case $c(M)$ and $c(\iota M)$ are identified by ι. Therefore, in counting the independent symmetry classes, it is enough to consider only one of the two standard forms.

2.1.2 The case where ιM is equivalent to M

This is the case where ιM and M are related by a change of basis, namely $c(\iota M) = c(M)$.

For example,

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{16}$$

In this case ι may act nontrivially on the factor systems belonging to $c(M)$. Put

$$\iota M = \sigma M, \quad \sigma = (\sigma_{ij})_{1 \leq i,j \leq n} \in GL_n(\mathbb{Z}_2). \tag{17}$$

It remains to compute how $\epsilon_{i,j}, \theta_1, \kappa_{1,j}$ transform under V. The change of representation matrices for the generators is given by

$$u_i \mapsto u'_i = \prod_{j=1}^{n} (u_j)^{\sigma_{ij}}. \tag{18}$$

- When $\phi_g = 0$ for all $g \in \mathbb{Z}_2^n$:

$$u'_i u'_k = \prod_{j=1}^{n} (u_j)^{\sigma_{ij}} \prod_{l=1}^{n} (u_l)^{\sigma_{kl}} = \prod_{j,l=1}^{n} (-1)^{\epsilon_{j,l}\sigma_{ij}\sigma_{kl}} u'_k u'_i. \tag{19}$$

  Thus the transformation is

$$\epsilon_{i,k} \mapsto \sum_{j,l=1}^{n} \epsilon_{j,l}\sigma_{ij}\sigma_{kl} = \sum_{j<l} \epsilon_{j,l}(\sigma_{ij}\sigma_{kl} + \sigma_{il}\sigma_{kj}) \quad \text{for } 1 \leq i < k \leq n. \tag{20}$$

  Equivalently,

$$\epsilon_{i,k} \mapsto (\sigma\epsilon\sigma^{\top})_{i,k} \quad \text{for } 1 \leq i < k \leq n. \tag{21}$$

  This is a $\mathbb{Z}_2$-linear action

$$\iota : H^2(\mathbb{Z}_2^n, U(1)) \to H^2(\mathbb{Z}_2^n, U(1)), \tag{22}$$

  so the orbits can be computed by the rank of the corresponding matrix. Writing the components as

$$\epsilon'_{i<k} = \sum_{j<l} [M_\iota]_{i<k,j<l}\epsilon_{j<l}, \tag{23}$$

  we have

$$[M_\iota]_{i<k,j<l} = \sigma_{ij}\sigma_{kl} + \sigma_{il}\sigma_{kj}. \tag{24}$$

  The number of independent factor systems is the number of orbits of the $\mathbb{Z}_2$ action ι, hence

$$|\{x \in H^2(\mathbb{Z}_2^n, U(1)) | \iota(x) = x\}| + \frac{1}{2}|\{x \in H^2(\mathbb{Z}_2^n, U(1)) | \iota(x) \neq x\}| \tag{25}$$

$$= |\{x \in H^2(\mathbb{Z}_2^n, U(1)) | \iota(x) = x\}| + \frac{1}{2}(|H^2(\mathbb{Z}_2^n, U(1))| - |\{x \in H^2(\mathbb{Z}_2^n, U(1)) | \iota(x) = x\}| \tag{26}$$

$$= \frac{1}{2}(|H^2(\mathbb{Z}_2^n, U(1))| + |\{x \in H^2(\mathbb{Z}_2^n, U(1)) | \iota(x) = x\}|) \tag{27}$$

$$= \frac{1}{2}(2^{n(n-1)/2} + 2^{|\mathrm{Ker}_{\mathbb{Z}_2}(I - M_\iota)|}). \tag{28}$$

- When there exists $g \in \mathbb{Z}_2^n$ such that $\phi_g = 1$:

  Take the standard form of M so that $\phi_1 = 1, \phi_{i>1} = 0$:

$$M = \begin{pmatrix} 1 & \eta_1 & c_1 \\ \mathbf{0} & \boldsymbol{\eta} & \boldsymbol{c} \end{pmatrix}. \tag{29}$$

  Then

$$\iota M = \begin{pmatrix} 1 & \eta_1 & c_1 \\ \mathbf{0} & \boldsymbol{\eta} & \boldsymbol{c} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \eta_1 & c_1 + \eta_1 + 1 \\ \mathbf{0} & \boldsymbol{\eta} & \boldsymbol{c} + \boldsymbol{\eta} \end{pmatrix}. \tag{30}$$

  By assumption, there exists Σ such that

$$\begin{pmatrix} 1 & \eta_1 & c_1 + \eta_1 + 1 \\ \mathbf{0} & \boldsymbol{\eta} & \boldsymbol{c} + \boldsymbol{\eta} \end{pmatrix} = \sigma \begin{pmatrix} 1 & \eta_1 & c_1 \\ \mathbf{0} & \boldsymbol{\eta} & \boldsymbol{c} \end{pmatrix}. \tag{31}$$

  Comparing both sides, σ has the form

$$\sigma = \begin{pmatrix} 1 & \boldsymbol{b}^\top \\ \mathbf{0} & \tilde{\sigma} \end{pmatrix} = (\sigma_{ij})_{1 \le i,j \le n}. \tag{32}$$

  The map of representation matrices satisfying the gauge fixing is

$$u_1' = u_1 (\prod_{j=2}^{n} (u_j)^{\sigma_{1j}})^* = (\prod_{j=2}^{n} (u_j)^{\sigma_{1j}}) u_1, \tag{33}$$

$$u_{i>1}' = (i)^{\sum_{j=2}^{n-1} \sum_{k=j+1}^{n} \epsilon_{j,k} \sigma_{ij} \sigma_{ik}} \prod_{j=2}^{n} (u_j)^{\sigma_{ij}}. \tag{34}$$

  Taking into account the gauge fixing $u_{i>1}^2 = 1$, the map of the invariant ϵ is computed as

$$u_1'(u_{i>1}')^* = (\prod_{j=2}^{n} (u_j)^{\sigma_{1j}}) u_1 \left((i)^{\sum_{j=2}^{n-1} \sum_{k=j+1}^{n} \epsilon_{j,k} \sigma_{ij} \sigma_{ik}} \prod_{k=2}^{n} (u_k)^{\sigma_{ik}} \right)^* \tag{35}$$

$$= (-i)^{\sum_{j=2}^{n-1} \sum_{k=j+1}^{n} \epsilon_{j,k} \sigma_{ij} \sigma_{ik}} \prod_{k=2}^{n} (-1)^{\epsilon_{1,k} \sigma_{ik}} (\prod_{j=2}^{n} (u_j)^{\sigma_{1j}}) (\prod_{k=2}^{n} (u_k)^{\sigma_{ik}}) u_1 \tag{36}$$

$$= (-1)^{\sum_{j=2}^{n-1} \sum_{k=j+1}^{n} \epsilon_{j,k} \sigma_{ij} \sigma_{ik}} \prod_{k=2}^{n} (-1)^{\epsilon_{1,k} \sigma_{ik}} \prod_{j,k=2}^{n} (-1)^{\epsilon_{j,k} \sigma_{1j} \sigma_{ik}} u_i' u_1', \tag{37}$$

$$u_{i>1}' u_{k>1}' = \prod_{j=2}^{n} (u_j)^{\sigma_{ij}} \prod_{l=2}^{n} (u_l)^{\sigma_{kl}} = \prod_{j,l=2}^{n} (-1)^{\epsilon_{j,l} \sigma_{ij} \sigma_{kl}} u_k' u_i'. \tag{38}$$

Thus, respectively,

$$\epsilon_{1,i>1} \mapsto \sum_{j=2}^{n-1}\sum_{k=j+1}^{n} \epsilon_{j,k}\sigma_{ij}\sigma_{ik} + \sum_{k=2}^{n}\epsilon_{1,k}\sigma_{ik} + \sum_{j,k=2}^{n}\epsilon_{j,k}\sigma_{1j}\sigma_{ik} \tag{39}$$

$$= \sum_{1<j<k}\epsilon_{j,k}\sigma_{ij}\sigma_{ik} + \sum_{k=2}^{n}\epsilon_{1,k}\sigma_{ik} + \sum_{1<j<k}\epsilon_{j,k}(\sigma_{1j}\sigma_{ik}+\sigma_{1k}\sigma_{ij}) \tag{40}$$

$$= \sum_{1<j<k}\epsilon_{j,k}(\sigma_{ij}\sigma_{ik}+\sigma_{1j}\sigma_{ik}+\sigma_{1k}\sigma_{ij}) + \sum_{1<k}\epsilon_{1,k}\sigma_{ik} \tag{41}$$

$$= \sum_{1<j<l}\epsilon_{j,l}(\sigma_{ij}\sigma_{il}+\sigma_{1j}\sigma_{il}+\sigma_{1l}\sigma_{ij}) + \sum_{1<k}\epsilon_{1,k}\sigma_{ik}, \tag{42}$$

$$\epsilon_{i,k} \mapsto \sum_{j,l=2}^{n}\epsilon_{j,l}\sigma_{ij}\sigma_{kl} \tag{43}$$

$$= \sum_{1<j<l}\epsilon_{j,l}(\sigma_{ij}\sigma_{kl}+\sigma_{il}\sigma_{kj}) \tag{44}$$

$$= \sum_{1<j<l}\epsilon_{j,l}(\sigma_{ij}\sigma_{kl}+\sigma_{il}\sigma_{kj}) \quad \text{for } 1<i<k\le n. \tag{45}$$

The map of the invariant θ_1 is obtained from

$$u'_1 u'^*_1 = (\prod_{j=2}^{n}(u_j)^{\sigma_{1j}})u_1\left((\prod_{k=2}^{n}(u_k)^{\sigma_{1k}})u_1\right)^* = (-1)^{\theta_1}\prod_{k=2}^{n}(-1)^{\epsilon_{1,k}\sigma_{1k}}\prod_{j,k=2,j<k}^{n}(-1)^{\epsilon_{j,k}\sigma_{1j}\sigma_{1k}}. \tag{46}$$

Therefore

$$\theta_1 \mapsto \theta_1 + \sum_{k=2}^{n}\epsilon_{1,k}\sigma_{1k} + \sum_{j=2}^{n-1}\sum_{k=j+1}^{n}\epsilon_{j,k}\sigma_{1j}\sigma_{1k} \tag{47}$$

$$= \sum_{1<j<l}\epsilon_{j,l}\sigma_{1j}\sigma_{1l} + \sum_{1<k}\epsilon_{1,k}\sigma_{1k} + \theta_1. \tag{48}$$

These define a $\mathbb{Z}_2$-linear action on $H^2(\mathbb{Z}_2^n, U(1))$, and the number of orbits of this action is the number of independent factor systems.

The transformation matrix is

$$\begin{pmatrix}\epsilon'_{1<i<k}\\ \epsilon'_{1<i}\\ \theta'_1\end{pmatrix} = \underbrace{\begin{pmatrix}\sigma_{ij}\sigma_{kl}+\sigma_{il}\sigma_{kj} & O & 0\\ \sigma_{ij}\sigma_{il}+\sigma_{1j}\sigma_{il}+\sigma_{1l}\sigma_{ij} & \sigma_{ik} & 0\\ \sigma_{1j}\sigma_{1l} & \sigma_{1k} & 1\end{pmatrix}}_{M_\iota}\begin{pmatrix}\epsilon_{1<j<l}\\ \epsilon_{1<k}\\ \theta_1\end{pmatrix}. \tag{49}$$

The number of independent factor systems is

$$= \frac{1}{2}(2^{n(n-1)/2+1} + 2^{|\mathrm{Ker}_{\mathbb{Z}_2}(I-M_\iota)|}). \tag{50}$$

The calculation results up to $p=10$ are shown in the tables.

References

[1] Kohei Kawabata, Ken Shiozaki, Masahito Ueda, Masatoshi Sato, "Symmetry and Topology in Non-Hermitian Physics", arXiv:1812.09133.

Table 1: $p = 0$ (entries are complex+real)

N	w/o factor system	w/ factor system	$H \to iH$
$N = 0$	1+0	1+0	1+0
$N = 1$	3+4	3+8	2+6
$N = 2$	1+6	2+24	2+15
$N = 3$	0+1	0+16	0+12
total	5+11 (= 16)	6+48 (= 54)	5+33 (= 38)

Table 2: $p = 1$ (entries are complex+real)

N	w/o factor system	w/ factor system	$H \to iH$
$N = 0$	1+0	1+0	1+0
$N = 1$	4+4	4+8	3+6
$N = 2$	4+10	8+40	6+27
$N = 3$	1+7	8+112	6+70
$N = 4$	0+1	0+128	0+80
total	10+22 (= 32)	21+288 (= 309)	16+183 (= 199)

Table 3: $p = 2$ (entries are complex+real)

N	w/o factor system	w/ factor system	$H \to iH$
$N = 0$	1+0	1+0	1+0
$N = 1$	4+4	4+8	3+6
$N = 2$	5+10	10+40	8+27
$N = 3$	4+11	32+176	22+118
$N = 4$	1+7	64+896	40+536
$N = 5$	0+1	0+2048	0+1152
total	15+33 (= 48)	111+3168 (= 3279)	74+1839 (= 1913)

Table 4: $p = 3$ (entries are complex+real)

N	w/o factor system	w/ factor system	$H \to iH$
$N = 0$	1+0	1+0	1+0
$N = 1$	4+4	4+8	3+6
$N = 2$	5+10	10+40	8+27
$N = 3$	5+11	40+176	30+118
$N = 4$	4+11	256+1408	168+920
$N = 5$	1+7	1024+14336	576+8384
$N = 6$	0+1	0+65536	0+34816
total	20+44 (= 64)	1335+81504 (= 82839)	786+44271 (= 45057)

Table 5: $p = 4$ (entries are complex+real)

N	w/o factor system	w/ factor system	$H \to iH$
$N = 0$	1+0	1+0	1+0
$N = 1$	4+4	4+8	3+6
$N = 2$	5+10	10+40	8+27
$N = 3$	5+11	40+176	30+118
$N = 4$	5+11	320+1408	232+920
$N = 5$	4+11	4096+22528	2624+14528
$N = 6$	1+7	32768+458752	17408+265216
$N = 7$	0+1	0+4194304	0+2162688
total	25+55 (= 80)	37239+4677216 (= 4714455)	20306+2443503 (= 2463809)

Table 6: $p = 5$ (entries are complex+real)

N	w/o factor system	w/ factor system	$H \to iH$
$N = 0$	1+0	1+0	1+0
$N = 1$	4+4	4+8	3+6
$N = 2$	5+10	10+40	8+27
$N = 3$	5+11	40+176	30+118
$N = 4$	5+11	320+1408	232+920
$N = 5$	5+11	5120+22528	3648+14528
$N = 6$	4+11	131072+720896	82944+461824
$N = 7$	1+7	2097152+29360128	1081344+16875520
$N = 8$	0+1	0+536870912	0+272629760
total	30+66 (= 96)	2233719+566976096 (= 569209815)	1168210+289982703 (= 291150913)

Table 7: $p = 6$ (entries are complex+real)

N	w/o factor system	w/ factor system	$H \to iH$
$N = 0$	1+0	1+0	1+0
$N = 1$	4+4	4+8	3+6
$N = 2$	5+10	10+40	8+27
$N = 3$	5+11	40+176	30+118
$N = 4$	5+11	320+1408	232+920
$N = 5$	5+11	5120+22528	3648+14528
$N = 6$	5+11	163840+720896	115712+461824
$N = 7$	4+11	8388608+46137344	5275648+29458432
$N = 8$	1+7	268435456+3758096384	136314880+2153775104
$N = 9$	0+1	0+137438953472	0+69256347648
total	35+77 (= 112)	276993399+141243932256 (= 141520925655)	141710162+71440058607 (= 71581768769)

Table 8: $p = 7$ (entries are complex+real)

N	w/o factor system	w/ factor system	$H \to iH$
$N = 0$	1+0	1+0	1+0
$N = 1$	4+4	4+8	3+6
$N = 2$	5+10	10+40	8+27
$N = 3$	5+11	40+176	30+118
$N = 4$	5+11	320+1408	232+920
$N = 5$	5+11	5120+22528	3648+14528
$N = 6$	5+11	163840+720896	115712+461824
$N = 7$	5+11	10485760+46137344	7372800+29458432
$N = 8$	4+11	1073741824+5905580032	673185792+3764387840
$N = 9$	1+7	68719476736+962072674304	34628173824+550561120256
$N = 10$	0+1	0+70368744177664	0+35321811042304
total	40+88 (= 128)	69803873655+71336769314400 (= 71406573188055)	35308852050+35876166486255 (= 35911475338305)

Table 9: $p = 8$ (entries are complex+real)

N	w/o factor system	w/ factor system	$H \to iH$
$N = 0$	1+0	1+0	1+0
$N = 1$	4+4	4+8	3+6
$N = 2$	5+10	10+40	8+27
$N = 3$	5+11	40+176	30+118
$N = 4$	5+11	320+1408	232+920
$N = 5$	5+11	5120+22528	3648+14528
$N = 6$	5+11	163840+720896	115712+461824
$N = 7$	5+11	10485760+46137344	7372800+29458432
$N = 8$	5+11	1342177280+5905580032	941621248+3764387840
$N = 9$	4+11	274877906944+1511828488192	172067127296+962877980672
$N = 10$	1+7	35184372088832+492581209243648	17660905521152+281681135140864
$N = 11$	0+1	0+72057594037927936	0+36099165763141632
total	45+99 (= 144)	35460602828151+72551693028122208 (= 72587153630950359)	17833921762130+36381813570586863 (= 36399647492348993)

Table 10: $p = 9$ (entries are complex+real)

N	w/o factor system	w/ factor system	$H \to iH$
$N = 0$	1+0	1+0	1+0
$N = 1$	4+4	4+8	3+6
$N = 2$	5+10	10+40	8+27
$N = 3$	5+11	40+176	30+118
$N = 4$	5+11	320+1408	232+920
$N = 5$	5+11	5120+22528	3648+14528
$N = 6$	5+11	163840+720896	115712+461824
$N = 7$	5+11	10485760+46137344	7372800+29458432
$N = 8$	5+11	1342177280+5905580032	941621248+3764387840
$N = 9$	5+11	343597383680+1511828488192	240786604032+962877980672
$N = 10$	4+11	140737488355328+774056185954304	88029649698816+492787367673856
$N = 11$	1+7	36028797018963968+504403158265495552	18049582881570816+288335929267978240
$N = 12$	0+1	0+147573952589676412928	0+73859033888876134400
total	50+110 (= 160)	36169879457535351+148079131321908813408 (= 148115301201366348759)	18137854266987346+74147863572184090863 (= 74166001426451078209)

Table 11: $p = 10$ (entries are complex+real)

N	w/o factor system	w/ factor system	$H \to iH$
$N = 0$	1+0	1+0	1+0
$N = 1$	4+4	4+8	3+6
$N = 2$	5+10	10+40	8+27
$N = 3$	5+11	40+176	30+118
$N = 4$	5+11	320+1408	232+920
$N = 5$	5+11	5120+22528	3648+14528
$N = 6$	5+11	163840+720896	115712+461824
$N = 7$	5+11	10485760+46137344	7372800+29458432
$N = 8$	5+11	1342177280+5905580032	941621248+3764387840
$N = 9$	5+11	343597383680+1511828488192	240786604032+962877980672
$N = 10$	5+11	175921860444160+774056185954304	123214021787648+492787367673856
$N = 11$	4+11	144115188075855872+792633534417207296	90107176919498752+504508711381762048
$N = 12$	1+7	73786976294838206464+1033017668127734890496	36929516944438067200+590403896749762543616
$N = 13$	0+1	0+604462909807314587353088	0+302379028856246970089472
total	55+121 (= 176)	73931267749724722551+605496720884550706355808 (= 605570652152300431078359)	37019747577115071314+302969937755462154373359 (= 303006957503039269444673)