Memo: Symmetry in Non-Hermitian Systems

Ken Shiozaki

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Abstract

One way to define symmetry in non-Hermitian systems is to define it so that, after Hermitianizing the Hamiltonian, it becomes the usual notion of symmetry for Hermitian systems [1]. This note records that convention.

1 Non-Hermitian Systems

Let G be a finite group and write its action on the parameter space X as gk. Let $\phi, \eta, c : G \to \mathbb{Z}/2 = \{\pm 1\}$ be homomorphisms. For a matrix M, introduce the notation

$$M^{\phi_g} = \begin{cases} M & (\phi_g = 1), \\ M^* & (\phi_g = -1), \end{cases} \tag{1.1}$$

and

$$M^{(\phi_g, \eta_g)} = \begin{cases} M & (\phi_g = 1, \eta_g = 1), \\ M^* & (\phi_g = -1, \eta_g = 1), \\ M^\top & (\phi_g = -1, \eta_g = -1), \\ M^\dagger & (\phi_g = 1, \eta_g = -1). \end{cases} \tag{1.2}$$

We consider a symmetry of the form

$$u_g H_k^{(\phi_g, \eta_g)} v_g^\dagger = c_g H_{gk}, \quad g \in G. \tag{1.3}$$

[1] Here u_g and v_g are unitary matrices. Then

$$c_{gh} H_{ghk} = \begin{cases} u_g (u_h H_k^{(\phi_h, \eta_h)} v_h^\dagger)^{\phi_g} v_g^\dagger & (\eta_g = 1), \\ u_g (v_h [H_k^{(\phi_h, \eta_h)}]^\dagger u_h^\dagger)^{\phi_g} v_g^\dagger & (\eta_g = -1) \end{cases} \tag{1.4}$$

holds. Let $z \in Z^2(G, U(1)_\phi)$ be a group cocycle and impose

$$z_{g,h} u_{gh} = \begin{cases} u_g u_h^{\phi_g} & (\eta_g = 1), \\ u_g v_h^{\phi_g} & (\eta_g = -1), \end{cases} \qquad z_{g,h} v_{gh} = \begin{cases} v_g v_h^{\phi_g} & (\eta_g = 1), \\ v_g u_h^{\phi_g} & (\eta_g = -1). \end{cases} \tag{1.5}$$

Then

$$c_{gh} H_{ghk} = u_{gh} H_{ghk}^{(\phi_{gh}, \eta_{gh})} v_{gh}^\dagger \tag{1.6}$$

is satisfied.[2] The factor systems of u and v are taken to be the same so that the phase factors cancel in the right-hand side of (1.4).

- In general, $z_{g,h}$ may depend on the point of the parameter space X.
- At a fixed point $gk = k$, the Hamiltonian H_k itself is an intertwiner between the "representations" u_g and v_g. Therefore, if a point-gapped Hamiltonian H_k exists at a fixed point, then u_g and v_g are "equivalent representations." Thus, when discussing point-gapped Hamiltonians, in addition to (1.5) one should choose u_g and v_g to be equivalent representations. See Ref. [1].

[1] Ref. [1] does not include complex conjugation, but the extension to cases with complex conjugation is straightforward.

[2] For example, if a transpose-type symmetry were instead defined with the transpose on the right-hand side, such as $u_g H_k u_g^\dagger = H_{gk}^\top$, then for two transpose-type symmetries g, h one would get $u_{gh} \sim u_g^* u_h$. It is better to define symmetry with complex conjugation, transposition, or Hermitian conjugation on the left-hand side, as in (1.3).

2 Hermitianization

Introduce the doubled Hermitian Hamiltonian

$$\tilde{H}_k = \begin{pmatrix} & H_k \\ H_k^\dagger & \end{pmatrix}_\sigma . \tag{2.1}$$

The doubled symmetry matrices are

$$\tilde{u}_g = \begin{cases} \begin{pmatrix} u_g & \\ & v_g \end{pmatrix} & (\eta_g = 1), \\ \begin{pmatrix} & u_g \\ v_g & \end{pmatrix} & (\eta_g = -1), \end{cases} \quad g \in G. \tag{2.2}$$

The symmetries of $\tilde{H}_k$ are summarized as

$$\sigma_z \tilde{H}_k \sigma_z = -\tilde{H}_k, \tag{2.3}$$

$$\tilde{u}_g \tilde{H}_k \tilde{u}_g^\dagger = c_g \tilde{H}_{gk}, \quad \tilde{u}_g \tilde{u}_h^{\phi_g} = z_{g,h} \tilde{u}_{gh}, \tag{2.4}$$

$$\sigma_z \tilde{u}_g = \eta_g \tilde{u}_g \sigma_z. \tag{2.5}$$

3 Real Line Gap

Suppose H_k satisfies the real-line-gap condition,

$$\Re(E_k) \neq 0, \quad E_k \in \mathrm{Spec}(H_k), \quad k \in X, \tag{3.1}$$

and, in addition,

$$u_g = v_g, \quad g \in G. \tag{3.2}$$

Then H_k can be deformed to a Hermitian Hamiltonian while preserving both the real line gap and the G symmetry.[3] The condition $H_k^\dagger = H_k$ is equivalent to requiring the doubled Hermitian Hamiltonian $\tilde{H}_k$ to have the additional chiral symmetry

$$\sigma_y \tilde{H}_k \sigma_y = -\tilde{H}_k \tag{3.3}$$

[2]. The relations between σ_y and the other symmetries are

$$\sigma_z \sigma_y = -\sigma_y \sigma_z, \tag{3.4}$$

$$\tilde{u}_g \sigma_y^{\phi_g} = \phi_g \eta_g \sigma_y \tilde{u}_g, \quad g \in G. \tag{3.5}$$

For the original Hamiltonian, using

$$\tilde{H}_k = H_k \sigma_x, \tag{3.6}$$

$$\tilde{u}_g = u_g \sigma_x^{(1-\eta_g)/2}, \quad g \in G, \tag{3.7}$$

one recovers

$$u_g H_k^{\phi_g} u_g^\dagger = c_g H_{gk}, \quad g \in G. \tag{3.8}$$

[3] When $u_g \neq v_g$, the projection $P_k = \frac{1}{2\pi i}\oint_C \frac{dz}{z-H_k}$ does not satisfy the G symmetry, so one cannot deform to a Hermitian Hamiltonian while preserving the G symmetry.

4 Imaginary Line Gap

Suppose H_k satisfies the imaginary-line-gap condition,

$$\Im(E_k) \neq 0, \quad E_k \in \mathrm{Spec}(H_k), \quad k \in X, \tag{4.1}$$

and, in addition,

$$u_g = v_g, \quad g \in G. \tag{4.2}$$

Then H_k can be deformed to an anti-Hermitian Hamiltonian while preserving the G symmetry and the imaginary line gap. The condition $H_k^\dagger = -H_k$ is equivalent to requiring the doubled Hermitian Hamiltonian $\tilde{H}_k$ to have the additional chiral symmetry

$$\sigma_x \tilde{H}_k \sigma_x = -\tilde{H}_k \tag{4.3}$$

[2]. The relations between σ_x and the other symmetries are

$$\sigma_z \sigma_x = -\sigma_x \sigma_z, \tag{4.4}$$

$$\tilde{u}_g \sigma_x^{\phi_g} = \sigma_x \tilde{u}_g, \quad g \in G. \tag{4.5}$$

For the original Hamiltonian, using

$$\tilde{H}_k = H_k i \sigma_y, \tag{4.6}$$

$$\tilde{u}_g = u_g \sigma_x^{(1-\eta_g)/2}, \quad g \in G, \tag{4.7}$$

one obtains

$$u_g H_k^{\phi_g} u_g^\dagger = \eta_g c_g H_{gk}, \quad g \in G. \tag{4.8}$$

Introducing the Hermitian Hamiltonian

$$H'_k = i H_k, \tag{4.9}$$

the symmetry becomes

$$u_g (H'_k)^{\phi_g} u_g^\dagger = c_g \eta_g \phi_g H'_{gk}, \quad g \in G. \tag{4.10}$$

References

[1] Ken Shiozaki and Seishiro Ono, *Symmetry indicator in non-Hermitian systems*, arXiv:2105.00677.

[2] Nobuyuki Okuma, Kohei Kawabata, Ken Shiozaki, and Masatoshi Sato, *Topological Origin of Non-Hermitian Skin Effects*, arXiv:1910.02878.