Compatibility of Flattening with Symmetry for Point-Gapped Hamiltonians

Ken Shiozaki

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1 Hermitian Systems

Let H_k be a finite-size Hermitian matrix. For a gapped Hamiltonian H_k, meaning that it has no zero eigenvalue for any k, write the spectral decomposition as

$$H_k = \sum_a \lambda_k^a P_k^a, \quad \lambda_k^a \neq 0. \tag{1}$$

Define the flattened Hamiltonian by

$$Q_k := P_k^+ - P_k^-, \quad P_k^+ = \sum_{a, \lambda_k^a > 0} P_k^a, \quad P_k^- = \sum_{a, \lambda_k^a < 0} P_k^a. \tag{2}$$

Let $u_k(g)$ be the unitary matrix implementing a symmetry transformation. We show that if

$$u_k(g) H_k u_k(g)^\dagger = H_{gk}, \quad g \in G, \tag{3}$$

then the flattened Hamiltonian Q_k satisfies the same symmetry.

Let C_+ and C_- be closed contours in the complex energy plane enclosing the positive and negative eigenvalues, respectively. Since H_k is a finite-dimensional matrix and is gapped, $C_\pm$ may be chosen independently of k. Then

$$P_k^\pm = \frac{1}{2\pi i} \oint_{C_\pm} \frac{dz}{z - H_k}. \tag{4}$$

Therefore

$$u_k(g) P_k^\pm u_k(g)^\dagger = \frac{1}{2\pi i} \oint_{C_\pm} \frac{dz}{z - u_k(g) H_k u_k(g)^\dagger} \tag{5}$$

$$= \frac{1}{2\pi i} \oint_{C_\pm} \frac{dz}{z - H_{gk}} = P_{gk}^\pm. \tag{6}$$

The claim follows.

The same argument also applies to more general Hermitian-system symmetries of the form

$$u_k(g) H_k^* u_k(g)^\dagger = H_{gk}, \tag{7}$$

$$u_k(g) H_k u_k(g)^\dagger = -H_{gk}, \tag{8}$$

$$u_k(g) H_k^* u_k(g)^\dagger = -H_{gk}. \tag{9}$$

Thus Q_k satisfies the corresponding symmetry as well.

2 Non-Hermitian Systems with a Point Gap

Let H_k be a finite-size square matrix, and assume a point gap, namely $\det H_k \neq 0$ for every k. For a singular-value decomposition

$$H_k = U_k \Sigma_k V_k^\dagger, \tag{10}$$

define the flattening, or unitarization, by

$$Q_k = U_k V_k^\dagger. \tag{11}$$

By uniqueness of the singular-value decomposition in this construction, Q_k is globally defined.

Let $u_k(g)$ and $v_k(g)$ be unitary matrices implementing a symmetry. Consider

$$u_k(g) H_k v_g(k)^\dagger = H_{gk}, \quad g \in G. \tag{12}$$

We show that Q_k obeys the same symmetry. Introduce the Hermitianized Hamiltonian

$$\tilde{H}_k = \begin{pmatrix} 0 & H_k \\ H_k^\dagger & 0 \end{pmatrix}. \tag{13}$$

This Hermitian Hamiltonian has chiral symmetry

$$\sigma_z \tilde{H}_k \sigma_z = -\tilde{H}_k, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{14}$$

and a G symmetry

$$\tilde{U}_k(g) \tilde{H}_k \tilde{U}_k(g)^\dagger = \tilde{H}_{gk}, \quad g \in G. \tag{15}$$

Hence the flattened Hermitian Hamiltonian $\tilde{Q}_k$ also respects the σ_z and $\tilde{U}_k(g)$ symmetries. A direct calculation gives

$$\tilde{Q}_k = \begin{pmatrix} 0 & U_k V_k^\dagger \\ V_k U_k^\dagger & 0 \end{pmatrix}. \tag{16}$$

This proves the claim.

The same conclusion holds for the more general non-Hermitian symmetries

$$u_k(g) H_k v_g(k)^\dagger = H_{gk}, \tag{17}$$
$$u_k(g) H_k^\dagger v_g(k)^\dagger = H_{gk}, \tag{18}$$
$$u_k(g) H_k^* v_g(k)^\dagger = H_{gk}, \tag{19}$$
$$u_k(g) H_k^\top v_g(k)^\dagger = H_{gk}, \tag{20}$$
$$u_k(g) H_k v_g(k)^\dagger = -H_{gk}, \tag{21}$$
$$u_k(g) H_k^\dagger v_g(k)^\dagger = -H_{gk}, \tag{22}$$
$$u_k(g) H_k^* v_g(k)^\dagger = -H_{gk}, \tag{23}$$
$$u_k(g) H_k^\top v_g(k)^\dagger = -H_{gk}, \tag{24}$$

because the Hermitianized Hamiltonian $\tilde{H}_k$ satisfies the corresponding Hermitian-system symmetry.