On $\lim_{N\to\infty} \mathrm{sp}(V_{\mathrm{OBC}}) \subset \mathrm{sp}(V_{\mathrm{SIBC}})$

Ken Shiozaki

YITP

June 1, 2026

Abstract

In this note, I give one explanation of $\lim_{N\to\infty} \mathrm{sp}(V_{\mathrm{OBC}}) \subset \mathrm{sp}(V_{\mathrm{SIBC}})$ using properties of Hermitian systems.

1 Outline of the explanation

Let V be a block Toeplitz operator of the form

$$V = \begin{pmatrix} A_0 & A_{-1} & A_{-2} & \cdots \\ A_1 & A_0 & A_{-1} & \\ A_2 & A_1 & A_0 & \\ \vdots & & & \ddots \end{pmatrix}. \tag{1}$$

Here the A_j are square matrices and are assumed to be sufficiently short-ranged. Correspondingly, define the block Toeplitz matrix

$$V_N = \begin{pmatrix} A_0 & A_{-1} & A_{-2} & \cdots & A_{-N+1} & A_{-N} \\ A_1 & A_0 & A_{-1} & & & A_{-N+1} \\ A_2 & A_1 & A_0 & & & \\ \vdots & & & \ddots & & \vdots \\ A_{N-1} & & & & A_0 & A_{-1} \\ A_N & A_{N-1} & \cdots & & A_1 & A_0 \end{pmatrix}. \tag{2}$$

What we want to show is the following lemma, corresponding to Lemma 11.1 of Ref. [1]:

$$\Lambda_w(V) \subset \mathrm{sp}(V). \tag{3}$$

Here $\Lambda_w(V) :=$ "$\limsup_{n\to\infty} \mathrm{sp}(V_n)$" is defined as the set of $\lambda \in \mathbb{C}$ for which there exists a sequence of eigenvalues λ_{n_j} of V_{n_j}, with $n_1 < n_2 < n_3 < \cdots$, such that $\lambda_{n_j} \to \lambda$. On the other hand, the spectrum of V is defined by

$$\mathrm{sp}(V) := \{\lambda \in \mathbb{C} \mid \lambda - V \text{ is not invertible}\}. \tag{4}$$

Looking at the proof of Lemma 11.1 of Ref. [1], the lemma (3) follows immediately from the following theorem, which corresponds to Theorem 3.7 (or part of it) of [1]:

$$V \text{ is invertible} \quad \Rightarrow \quad \limsup_{n\to\infty} ||V_n^{-1}||_p < \infty. \tag{5}$$

Here $\limsup_{n\to\infty}$ is the limit superior

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \sup_{k\geq n} a_k, \tag{6}$$

and the norm statement holds for $1 \leq p \leq \infty$.[1] In particular, in the $p = 2$ norm, the statement $\limsup_{n\to\infty} ||V_n^{-1}||_2 < \infty$ can equivalently be phrased as saying that, for sufficiently large n, the smallest singular value of V_n is bounded from below, $\sigma_{\min}(V_n) > \delta > 0$, since $||V_n^{-1}||_2 = \sigma_{\max}(V_n^{-1}) = \sigma_{\min}(V_n)^{-1}$.

Indeed, assuming (5), (3) is proved as follows. Let $\lambda_0 \notin \mathrm{sp}(V)$. By (5), there is a sufficiently large n_0 such that, for $n \geq n_0$, one can take $||(V_n - \lambda_0)^{-1}|| < M < \infty$. Then for $\lambda \in \mathbb{C}$ satisfying $|\lambda - \lambda_0| < \frac{1}{2M}$, there is no eigenvalue of V_n that approaches λ when $n \geq n_0$. Indeed,

$$\begin{aligned}
||(V_n - \lambda)^{-1}|| &= ||(V_n - \lambda_0 + \lambda_0 - \lambda)^{-1}|| && (8)\\
&= ||(V_n - \lambda_0)^{-1}(1 - (\lambda - \lambda_0)(V_n - \lambda_0)^{-1})^{-1}|| && (9)\\
&\leq ||(V_n - \lambda_0)^{-1}|| \times ||(1 - (\lambda - \lambda_0)(V_n - \lambda_0)^{-1})^{-1}|| && (10)\\
&\leq ||(V_n - \lambda_0)^{-1}|| \times ||\sum_{k=0}^{\infty}\{(\lambda - \lambda_0)(V_n - \lambda_0)^{-1}\}^k|| && (11)\\
&\leq ||(V_n - \lambda_0)^{-1}|| \times \sum_{k=0}^{\infty}\{|\lambda - \lambda_0| \times ||(V_n - \lambda_0)^{-1}||\}^k && (12)\\
&= ||(V_n - \lambda_0)^{-1}|| \times \frac{1}{1 - |\lambda - \lambda_0| \times ||(V_n - \lambda_0)^{-1}||} && (13)\\
&\leq M \times \frac{1}{1 - \frac{1}{2}} = 2M. && (14)
\end{aligned}$$

Thus eigenvalues of V_n do not exist within distance $1/M$ of λ_0. Hence $\lambda_0 \notin \Lambda_w(V)$. Notice that this argument does not depend on the choice of norm.

Let us "prove" Theorem (5) in the case $p = 2$. It suffices to show that, when V is invertible, the smallest singular value of V_n has a "gap" for sufficiently large n. Double the degrees of freedom and introduce the Hermitian Hamiltonians

$$H = \begin{pmatrix} & V \\ V^\dagger & \end{pmatrix}, \qquad H_n = \begin{pmatrix} & V_n \\ V_n^\dagger & \end{pmatrix}. \tag{15}$$

[1] Memo:

$$||A||_p := \sup_{x\neq 0} \frac{||Ax||_p}{||x||_p}, \quad ||x||_p = \left(\sum_n |x_n|^p\right)^{1/p}, \quad ||x||_\infty = \sup_n |x_n|.$$

In particular, for $p = 2$,

$$||A||_2 = \sqrt{\lambda_{\max}(A^\dagger A)} = \sigma_{\max}(A), \tag{7}$$

that is, it is equal to the largest singular value of A.

Note that the eigenvalues of the OBC Hamiltonian H_n are precisely the singular values $\pm\sigma_j$ of V_n. Now assume that

$$V \text{ is invertible} \quad \Rightarrow \quad H \text{ has a finite energy gap } E_g. \tag{16}$$

For a Toeplitz operator $V = T(a)$, one can show that invertibility of $T(a)$ is equivalent to $a(t) \neq 0$ for all $t \in \mathbb{T}$ (no gap closing) together with $\operatorname{wind} a = 0$ (no zero state of H). Thus (16) holds, and the converse also holds. For a general block Toeplitz operator, one should carefully check whether this is true. (See the next subsection.) Furthermore, make the following "physical" assumption for Hermitian systems:

$$\begin{aligned}&\text{If } H \text{ has a finite energy gap } E_g,\\ &\text{then, once } n \text{ is sufficiently large compared with } E_g^{-1},\ H_n \text{ also has a finite energy gap } E_g.\end{aligned} \tag{17}$$

Then, for sufficiently large n, the singular values of V_n are bounded from below:

$$\sigma_j(V_n) \geq E_g. \tag{18}$$

This gives (5) for $p = 2$.

1.1 Invertibility of V and the spectrum of H

We want to show that, when V is invertible, H has a finite energy gap $E_g > 0$. For the spectrum of V, one has

$$V(\mathbb{T}) \subset \operatorname{sp}(V). \tag{19}$$

Therefore, if V is invertible, then $0 \notin V(\mathbb{T})$. Since

$$\det H(k) = |\det V(k)|^2, \tag{20}$$

this means that the bulk of H is gapped. For the spectrum of H, if we use the "common wisdom for Hermitian systems," then when the bulk of H has an energy gap, the spectrum separates as

$$\operatorname{sp}(H) = H(\mathbb{T}) \cup \{ \text{ edge states } \}. \tag{21}$$

Thus it suffices to show that there are no zero-energy edge states. The contraposition

$$H \text{ has a zero-energy edge state} \quad \Rightarrow \quad V \text{ is not invertible} \tag{22}$$

can be shown as follows. A zero-energy state of H is localized at the edge and can be labeled by chirality. In the case

$$\begin{pmatrix} & V \\ V^\dagger & \end{pmatrix} \begin{pmatrix} 0 \\ |0\rangle \end{pmatrix} = 0, \tag{23}$$

we have $V|0\rangle = 0$, so V has a localized zero-energy state and is not invertible. In the case

$$\begin{pmatrix} & V \\ V^\dagger & \end{pmatrix} \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} = 0, \tag{24}$$

we have $V^\dagger|0\rangle = 0$, so again V is not invertible.

Thus, when V is invertible, the bulk $H(\mathbb{T})$ of H has a finite energy gap, and H has no zero-energy edge state.

1.2 Summary

The above discussion can be summarized briefly as follows. We want to show

$$\Lambda_w(V) \subset \mathrm{sp}(V). \tag{25}$$

Assume $\lambda_0 \notin \mathrm{sp}(V)$. Then the corresponding Hermitian Hamiltonian

$$H(\lambda_0) = \begin{pmatrix} & V-\lambda_0 \\ V^\dagger - \lambda_0^* & \end{pmatrix} \tag{26}$$

has a finite energy gap $E_g(\lambda_0)$. Then, by the "common wisdom for Hermitian systems," the OBC Hamiltonian

$$H_n(\lambda_0) = \begin{pmatrix} & V_n-\lambda_0 \\ V_n^\dagger - \lambda_0^* & \end{pmatrix} \tag{27}$$

should also have a finite energy gap $E_g(\lambda_0)$, provided n is sufficiently large compared with the bulk gap. For the finite matrix $H_n(\lambda_0)$, note that the positive eigenvalues of $H_n(\lambda_0)$ are equal to the singular values of $V_n - \lambda_0$. By the Weyl inequality, any eigenvalue $\lambda_j - \lambda_0$ of $V_n - \lambda_0$ satisfies

$$|\lambda_j - \lambda_0| \geq \sigma_{\min}(H_n(\lambda_0)). \tag{28}$$

Therefore, if n is sufficiently large,

$$|\lambda_j - \lambda_0| \geq \sigma_{\min}(H_n(\lambda_0)) \geq E_g(\lambda_0) > 0. \tag{29}$$

It follows that there is no eigenvalue of V_n approaching λ_0 as $n \to \infty$. Thus $\lambda_0 \notin \Lambda_w(V)$.

A Relation between eigenvalues and singular values

In general, the absolute values of the eigenvalues of a matrix can be bounded from below by the smallest singular value. List the eigenvalues λ_j of a matrix A in decreasing order of absolute value, $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_L|$, and similarly list the singular values of A as $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_L$. Noting the Weyl inequalities

$$\prod_{j=1}^{k} |\lambda_j| \leq \prod_{j=1}^{k} \sigma_j, \quad (k = 1, \ldots, L-1), \tag{30}$$

$$\prod_{j=1}^{L} |\lambda_j| = \prod_{j=1}^{L} \sigma_j, \tag{31}$$

we obtain

$$|\lambda_L| = \frac{\prod_{j=1}^{L} \sigma_j}{\prod_{j=1}^{L-1} |\lambda_j|} \geq \frac{\prod_{j=1}^{L} \sigma_j}{\prod_{j=1}^{L-1} \sigma_j} = \sigma_L. \tag{32}$$

Taking the $k = 1$ Weyl inequality into account, we have, in the end,

$$\sigma_1 \geq |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_L| \geq \sigma_L. \tag{33}$$

References

[1] Böttcher, Albrecht, and Sergei M. Grudsky. *Spectral properties of banded Toeplitz matrices*, **96**, SIAM, 2005.