The Polyakov–Wiegmann Formula for Non-Simply-Connected Groups

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Abstract

When the group G is not simply connected, the Polyakov–Wiegmann (PW) formula receives a correction [1]. This note follows the proof of that correction.

1 The PW Formula

We consider $G = U(n)$ and $G = SO(n)$. Let M be a closed oriented two dimensional manifold. For a smooth map

$$g : M \to G, \tag{1.1}$$

choose an extension to a three dimensional manifold X,

$$\tilde{g} : X \to G, \quad \partial X = M, \quad \tilde{g}|_M = g. \tag{1.2}$$

[1] Define the Wess–Zumino term by

$$H(g) = \frac{1}{12\pi} \mathrm{tr}\,[(g^{-1}dg)^3], \tag{1.3}$$

$$\mathrm{WZ}[g] := \int_X H(\tilde{g}) \in \begin{cases} \mathbb{R}/2\pi\mathbb{Z} & (G = U(n)), \\ \mathbb{R}/4\pi\mathbb{Z} & (G = SO(n)). \end{cases} \tag{1.4}$$

For a homotopy of maps $g_t : M \to G$, $t \in [0,1]$, one has

$$\mathrm{WZ}[g_1] = \mathrm{WZ}[g_0] + \int_{M\times[0,1]} H(g_t). \tag{1.5}$$

Furthermore, since $\mathrm{tr}\,[(g^\dagger dg)^3]$ does not depend on the metric of X, $\mathrm{WZ}[g]$ is invariant under changes of the metric on M. We want to compute

- $\mathrm{WZ}[g_0 g_1]$,
- $\mathrm{WZ}[g_0 \oplus g_1]$.

Even if $\tilde{g}_0$ and $\tilde{g}_1$ are extensions of g_0 and g_1, respectively, the product $\tilde{g}_0\tilde{g}_1$ is not generally an extension of $g_0 g_1$ when the genus of M is nonzero. The same issue appears for direct sums. This happens when the homotopy group $\pi_1(G)$ along cycles in $H_1(M)$ for the maps g_0, g_1 is nontrivial. For instance, on a torus, if g_0 winds in the x direction and can be contracted in the y direction, whereas g_1 winds in the y direction and can be contracted in the x direction, then $g_0 g_1$ cannot be contracted in both the x and y directions.

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[1] For the existence of such an extension, one should examine $\Omega_2^{\mathrm{SO}}(G)$. Since $\Omega_0^{\mathrm{SO}} = \mathbb{Z}$ and $\Omega_{i=1,2}^{\mathrm{SO}} = 0$, one has $\Omega_2^{\mathrm{SO}}(G) \cong H_2(G,\mathbb{Z})$.

A direct computation gives

$$H(g_0 g_1) = H(g_0) + H(g_1) + d\,\mathrm{tr}\left[\frac{1}{4\pi} g_0^{-1} dg_0\, g_1 dg_1^{-1}\right]. \tag{1.6}$$

Thus one expects an almost-standard Polyakov–Wiegmann expression

$$\mathrm{WZ}[g_0 g_1] \sim \mathrm{WZ}[g_0] + \mathrm{WZ}[g_1] + \frac{1}{4\pi}\int_M \mathrm{tr}\,[g_0^{-1} dg_0\, g_1 dg_1^{-1}]. \tag{1.7}$$

Define

$$c[g,h] := \mathrm{WZ}[gh] - \mathrm{WZ}[g] - \mathrm{WZ}[h] - \frac{1}{4\pi}\int_M \mathrm{tr}\,[g^{-1} dg\, h\, dh^{-1}]. \tag{1.8}$$

Then:

- $c[g,h]$ depends only on the homotopy classes of g and h.

Indeed, if g_t is a homotopy, then $g_t h$ is also a homotopy, and

$$c[g_1,h] - c[g_0,h] \tag{1.9}$$

$$= \int_{M\times[0,1]} H(g_t h) - \int_{M\times[0,1]} H(g_t) - \frac{1}{4\pi}\int_M \mathrm{tr}\,[g_1^{-1} dg_1\, h\, dh^{-1}] + \frac{1}{4\pi}\int_M \mathrm{tr}\,[g_0^{-1} dg_0\, h\, dh^{-1}] \tag{1.10}$$

$$= \int_{M\times[0,1]} H(h) + \frac{1}{4\pi}\int_{M\times[0,1]} d\,\mathrm{tr}\,[g_t^{-1} dg_t\, h\, dh^{-1}] \tag{1.11}$$

$$- \frac{1}{4\pi}\int_M \mathrm{tr}\,[g_1^{-1} dg_1\, h\, dh^{-1}] + \frac{1}{4\pi}\int_M \mathrm{tr}\,[g_0^{-1} dg_0\, h\, dh^{-1}] \tag{1.12}$$

$$= 0. \tag{1.13}$$

The argument for h is the same. Hence we write

$$c([g],[h]) = c[g,h]. \tag{1.14}$$

Since it depends only on homotopy classes, it is a sum of contributions from the holes of the Riemann surface, and it is enough to compute it for $M = T^2$. For $G = U(n), SO(n)$, since $\pi_2(G) = 0$, a homotopy class on the torus is fixed by specifying the homotopy classes on the two cycles of the torus.

First consider $U(n)$. Let

$$g = \begin{pmatrix} e^{i(n_x x + n_y y)} & \\ & 1_{n-1} \end{pmatrix}, \quad h = \begin{pmatrix} e^{i(m_x x + m_y y)} & \\ & 1_{n-1} \end{pmatrix}. \tag{1.15}$$

The maps g, h, gh are independent of one direction on the torus after choosing representatives, so

$$\mathrm{WZ}[g] = \mathrm{WZ}[h] = \mathrm{WZ}[gh] = 0. \tag{1.16}$$

Therefore

$$c([g],[h]) = -\frac{1}{4\pi}\int_{T^2} \mathrm{tr}\,[g^{-1} dg\, h\, dh^{-1}] \tag{1.17}$$

$$= -\frac{1}{4\pi}\int_{T^2} (n_x m_y - n_y m_x)\, dx\, dy \tag{1.18}$$

$$= -\pi(n_x m_y - n_y m_x) \in \mathbb{R}/2\pi\mathbb{Z}. \tag{1.19}$$

For $SO(n \geq 2)$, take

$$g = \begin{pmatrix} e^{i(n_x \sigma_y x + n_y \sigma_y y)} & \\ & 1_{n-2} \end{pmatrix}, \quad h = \begin{pmatrix} e^{i(m_x \sigma_y x + m_y \sigma_y y)} & \\ & 1_{n-2} \end{pmatrix}. \tag{1.20}$$

Then

$$c([g],[h]) = -\frac{1}{4\pi}\int_{T^2} \operatorname{tr}[g^{-1}dg\, h\, dh^{-1}] \tag{1.21}$$

$$= -\frac{1}{4\pi}\int_{T^2} 2(n_x m_y - n_y m_x)\, dx\, dy \tag{1.22}$$

$$= -2\pi(n_x m_y - n_y m_x) \in \mathbb{R}/4\pi\mathbb{Z}. \tag{1.23}$$

In both cases, the correction to the PW formula depends only on the winding numbers of g and h and is $\mathbb{Z}_2$-valued.

For a direct sum, after adding a trivial block to match the sizes of g and h,

$$\mathrm{WZ}[g \oplus h] = \mathrm{WZ}\left[\begin{pmatrix} gh^{-1} & \\ & 1 \end{pmatrix}\begin{pmatrix} h & \\ & h \end{pmatrix}\right] \tag{1.24}$$

$$= \mathrm{WZ}\left[\begin{pmatrix} gh^{-1} & \\ & 1 \end{pmatrix}\right] + \mathrm{WZ}\left[\begin{pmatrix} h & \\ & h \end{pmatrix}\right] + \frac{1}{4\pi}\int_M \operatorname{tr}[(gh^{-1})^{-1}d(gh^{-1})\, h\, dh^{-1}] + c([gh^{-1}], 2[h]) \tag{1.25}$$

$$= \mathrm{WZ}[gh^{-1}] + 2\mathrm{WZ}[h] + \frac{1}{4\pi}\int_M \operatorname{tr}[(gh^{-1})^{-1}d(gh^{-1})\, h\, dh^{-1}] \tag{1.26}$$

$$= \mathrm{WZ}[g] + \mathrm{WZ}[h] + \frac{1}{4\pi}\int_M \operatorname{tr}[g^{-1}dg\, h^{-1}dh] + c([g],[h^{-1}]) \tag{1.27}$$

$$+ \frac{1}{4\pi}\int_M \operatorname{tr}[(gh^{-1})^{-1}d(gh^{-1})\, h\, dh^{-1}] \tag{1.28}$$

$$= \mathrm{WZ}[g] + \mathrm{WZ}[h] - c([g],[h]). \tag{1.29}$$

Thus the same type of correction term appears for direct sums.

References

[1] Krzysztof Gawedzki and Konrad Waldorf, *Polyakov–Wiegmann Formula and Multiplicative Gerbes*, arXiv:0908.1130.