# The Pseudoinverse Matrix

Ken Shiozaki

June 1, 2026

## 1 Moore–Penrose inverse

The following discussion is based on the Wikipedia article [1]. Let $A$ be an $m \times n$ matrix. A matrix $A^+$ satisfying the following four conditions is called the Moore–Penrose inverse:

1. $AA^+A = A$.
2. $A^+AA^+ = A^+$.
3. $AA^+$ is Hermitian: $(AA^+)^\dagger = AA^+$.
4. $A^+A$ is Hermitian: $(A^+A)^\dagger = A^+A$.

Take a singular value decomposition of $A$, and let $r = \mathrm{rank}(A)$:

$$A = U \begin{pmatrix} \Sigma & O \\ O & O \end{pmatrix} V^\dagger. \tag{1}$$

Here $U$ and $V$ are $m \times m$ and $n \times n$ unitary matrices, respectively, and

$$\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_r), \quad \sigma_1 \geq \cdots \geq \sigma_r > 0 \tag{2}$$

is the $r \times r$ diagonal matrix of the nonzero singular values of $A$. The existence of a matrix satisfying the four conditions is seen by setting

$$A^+ = V \begin{pmatrix} \Sigma^{-1} & O \\ O & O \end{pmatrix} U^\dagger. \tag{3}$$

Then

$$AA^+ = U \begin{pmatrix} 1 & O \\ O & O \end{pmatrix} U^\dagger, \tag{4}$$

$$A^+A = V \begin{pmatrix} 1 & O \\ O & O \end{pmatrix} V^\dagger, \tag{5}$$

so this $A^+$ satisfies all four conditions. This expression also shows that $AA^+$ and $A^+A$ are orthogonal projections.

It remains to prove uniqueness. Let $\tilde{A}^+$ be another solution. From conditions 3 and 4,

$$\begin{aligned} AA^+ &= A\tilde{A}^+AA^+ = (A\tilde{A}^+)^\dagger (AA^+)^\dagger = \tilde{A}^{+\dagger}A^\dagger A^{+\dagger}A^\dagger \\ &= \tilde{A}^{+\dagger}(AA^+A)^\dagger = \tilde{A}^{+\dagger}A^\dagger = (A\tilde{A}^+)^\dagger = A\tilde{A}^+. \end{aligned} \tag{6}$$

Similarly,

$$\begin{aligned} A^+A &= \tilde{A}^+A\tilde{A}^+A = (\tilde{A}^+A)^\dagger(\tilde{A}^+A)^\dagger = A^\dagger\tilde{A}^{+\dagger}A^\dagger\tilde{A}^{+\dagger} \\ &= (A\tilde{A}^+A)^\dagger\tilde{A}^{+\dagger} = A^\dagger\tilde{A}^{+\dagger} = (\tilde{A}^+A)^\dagger = \tilde{A}^+A. \end{aligned} \tag{7}$$

Therefore

$$A^+ = A^+AA^+ = A^+A\tilde{A}^+ = \tilde{A}^+A\tilde{A}^+ = \tilde{A}^+. \tag{8}$$

The projection properties are summarized as follows. Write the linear map represented by $A$ as

$$f : V \to W, \quad f(a_1, \dots, a_n) = (b_1, \dots, b_m)A. \tag{9}$$

Then the singular value decomposition gives

$$f(a_1, \dots, a_n)V = (b_1, \dots, b_m)U \begin{pmatrix} \Sigma & O \\ O & O \end{pmatrix}. \tag{10}$$

From this expression one sees that

- $AA^+$ is the orthogonal projection onto $\mathrm{Im}\, A$;
- $1_m - AA^+$ is the orthogonal projection onto $(\mathrm{Im}\, A)^\perp$;
- $A^+A$ is the orthogonal projection onto $(\mathrm{Ker}\, A)^\perp$;
- $1_n - A^+A$ is the orthogonal projection onto $\mathrm{Ker}\, A$.

## 2 Solutions of simultaneous linear equations

Let $A$ be an $m \times n$ complex matrix and consider

$$Ax = b, \quad x \in \mathbb{C}^n, \quad b \in \mathbb{C}^m. \tag{11}$$

A solution need not exist. The necessary and sufficient condition for a solution is $b \in \mathrm{Im}\, A$. In terms of the orthogonal projection, this condition is

$$AA^+b = b. \tag{12}$$

When this holds,

$$x = A^+b \tag{13}$$

is one solution. The solution is not unique; there remains the freedom to add an arbitrary vector in $\mathrm{Ker}\, A$. The general solution is

$$x = A^+b + (1_n - A^+A)w, \tag{14}$$

where $w \in \mathbb{C}^n$ is arbitrary.

## References

[1] https://en.wikipedia.org/wiki/Moore%E2%80%93Penrose_inverse