Adiabatic Pumps and Berry Phases in One-Dimensional Spin Systems with $\mathbb{Z}_2$ Symmetry

Ken Shiozaki

June 1, 2026

Abstract

This is a calculation note on the Berry phase in an adiabatic pump of a spin chain with $\mathbb{Z}_2$ symmetry.

1 Spin system and $\mathbb{Z}_2$ symmetry

The model is [1]

$$H_\theta^\sigma = -\sum_{j=1}^{L} B_j^\theta, \quad B_j^\theta = U_\theta \sigma_x U_\theta^\dagger, \tag{1}$$

where the local unitary transformation depends on the boundary condition:

$$U_\theta = e^{\frac{i\theta}{2} N_{\rm dw}}, \tag{2}$$

$$N_{\rm dw} = \begin{cases} \sum_{j=1}^{L-1} \frac{1-\sigma_j^z\sigma_{j+1}^z}{2} + \frac{1-\sigma_L^z\sigma_1^z}{2} & \text{(untwisted)}, \\ \sum_{j=1}^{L-1} \frac{1-\sigma_j^z\sigma_{j+1}^z}{2} + \frac{1+\sigma_L^z\sigma_1^z}{2} & \text{(twisted)}. \end{cases} \tag{3}$$

Note that

$$U_{2\pi} = \begin{cases} \mathrm{Id} & \text{(untwisted)}, \\ -\mathrm{Id} & \text{(twisted)}. \end{cases} \tag{4}$$

The $\mathbb{Z}_2$ symmetry is

$$V = \prod_j \sigma_j^x. \tag{5}$$

The normalized ground state is

$$|\Psi_\theta\rangle = U_\theta |\Psi_0\rangle, \quad |\Psi_0\rangle = \frac{1}{2^{L/2}} \sum_{\sigma_1,\dots,\sigma_L} |\sigma_1\sigma_2\cdots\rangle. \tag{6}$$

Because of the periodicity of U_θ,

$$|\Psi_{2\pi}\rangle = \begin{cases} |\Psi_0\rangle & \text{(untwisted)}, \\ -|\Psi_0\rangle & \text{(twisted)}. \end{cases} \tag{7}$$

The phase coming from the patch transformation contributes to the Berry phase. There is also a contribution from the connection:

$$\exp \int_0^{2\pi} \langle \Psi_\theta | d\Psi_\theta \rangle = \exp \int_0^{2\pi} \langle \Psi_0 | U_\theta^\dagger dU_\theta | \Psi_0 \rangle \tag{8}$$

$$= e^{\pi i \langle \Psi_0 | N_{\mathrm{dw}} | \Psi_0 \rangle}. \tag{9}$$

Thus it suffices to compute the expectation value of the domain-wall number operator N_{dw} in the ground state $|\Psi_0\rangle$. By translation symmetry and the $\mathbb{Z}_2$ gauge symmetry $\sigma_j^x |\Psi_0\rangle = |\Psi_0\rangle$,

$$\langle \Psi_0 | \sigma_j^z \sigma_{j+1}^z | \Psi_0 \rangle = 0. \tag{10}$$

Hence the expectation value of a domain wall on each link is $1/2$, independently of the boundary condition, and

$$\langle \Psi_0 | N_{\mathrm{dw}} | \Psi_0 \rangle = \frac{L}{2}. \tag{11}$$

Combining this with the patch-transformation contribution, the Berry phase is

$$e^{i\gamma} = \begin{cases} i^L & \text{(untwisted)}, \\ -i^L & \text{(twisted)}, \end{cases} \tag{12}$$

so the ratio is quantized to -1.

In general, the ratio of Berry phases is not quantized at finite L, but it is expected to become quantized in the limit $L \to \infty$. An example of a fermionic system where it is not quantized was given in [2]. Here we compute the corresponding spin model. For $r > 0$, consider, up to normalization,

$$|\Psi_0\rangle = \frac{1}{\sqrt{N(r)}} \sum_{\sigma_1,\dots,\sigma_L} r^{N_{\mathrm{dw}}/2} |\sigma_1 \sigma_2 \cdots \rangle. \tag{13}$$

The normalization constant is

$$N(r) = \sum_{\sigma_1,\dots,\sigma_L} \langle \sigma_1 \sigma_2 \cdots | r^{N_{\mathrm{dw}}} | \sigma_1 \sigma_2 \cdots \rangle. \tag{14}$$

There are $\binom{L}{l}$ ways to place l domain walls on L links, and a factor of 2 from the global spin flip. With periodic boundary conditions only an even number of domain walls appears, whereas with twisted boundary conditions only an odd number appears. Hence

$$N(r) = \sum_{l \in \mathrm{even/odd}} 2 \binom{L}{l} r^l = (1+r)^L \pm (1-r)^L. \tag{15}$$

The expectation value of the number of domain walls is

$$\langle \Psi_0 | N_{\mathrm{dw}} | \Psi_0 \rangle = \frac{r N'(r)}{N(r)} = rL \frac{(1+r)^{L-1} \mp (1-r)^{L-1}}{(1+r)^L \pm (1-r)^L}. \tag{16}$$

Therefore the difference of Berry phases between twisted and untwisted boundary conditions is

$$\gamma_{\mathrm{twisted}} - \gamma_{\mathrm{untwisted}} = \pi + \pi r L \left[\frac{(1+r)^{L-1} + (1-r)^{L-1}}{(1+r)^L - (1-r)^L} - \frac{(1+r)^{L-1} - (1-r)^{L-1}}{(1+r)^L + (1-r)^L} \right] \tag{17}$$

$$= \pi + \frac{\pi r L}{1-r^2} \frac{1}{\left(\frac{1+r}{1-r}\right)^L - \left(\frac{1-r}{1+r}\right)^L}. \tag{18}$$

Thus it approaches π exponentially fast in L.

References

[1] K. Shiozaki, arXiv:2110.10665.

[2] S. Ohyama, K. Shiozaki, and M. Sato, arXiv:2206.01110.