Spin-Chain Models on $RP^2 \times S^1$ and $L(3,1) \times S^1$

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1 The $RP^2 \times S^1$ Model

I want to construct a cluster model on $RP^2 \times S^1$. For domain walls of the σ spins, first construct a decorated-domain-wall (DDW) state satisfying the following conditions:

- for $\uparrow\uparrow$ and $\downarrow\downarrow$, use a $\mathbb{Z}_2$-nontrivial wave function on RP^2;
- for $\uparrow\downarrow$ and $\downarrow\uparrow$, use a $\mathbb{Z}_2$-trivial wave function on RP^2.

A simple representative of the $\mathbb{Z}_2$-nontrivial wave function on RP^2 is

$$|u(\boldsymbol{n})\rangle = \begin{pmatrix} n_z \\ n_x + i n_y \end{pmatrix}, \quad \boldsymbol{n} \in S^2. \tag{1}$$

Since

$$|-\boldsymbol{n}\rangle = -|\boldsymbol{n}\rangle, \tag{2}$$

the state $|\boldsymbol{n}\rangle$ is a state on $RP^2 = S^2/\mathbb{Z}_2$. Moreover, computing the first Chern class $c_1 \in H^2(RP^2, \mathbb{Z}) = \mathbb{Z}_2$ of $|\boldsymbol{n}\rangle$, one finds that it is nontrivial [1]. Note that the Berry curvature computed from $|n\rangle$ is nonzero, and the Berry connection is not flat. On the other hand, a $\mathbb{Z}_2$-trivial wave function on RP^2 is, for example, the RP^2-independent state

$$|\mathrm{triv}(\boldsymbol{n})\rangle = (1). \tag{3}$$

Now implement this construction. Place spin-1/2 degrees of freedom on sites, and introduce on each bond a three-dimensional Hilbert space describing the RP^2-nontrivial and trivial states. As a basis of the bond Hilbert space, consider the following $\boldsymbol{n}$-dependent basis:

$$(|u(\boldsymbol{n})\rangle, |v(\boldsymbol{n})\rangle, |w(\boldsymbol{n})\rangle) = \begin{pmatrix} n_z & n_x - i n_y & 0 \\ n_x + i n_y & -n_z & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

This basis is obtained from the standard basis by the following unitary transformation:

$$(|u(\boldsymbol{n})\rangle, |v(\boldsymbol{n})\rangle, |w(\boldsymbol{n})\rangle) = U(\boldsymbol{n}) \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad U(\boldsymbol{n}) := \begin{pmatrix} \boldsymbol{n}\cdot\boldsymbol{\tau} & \\ & 1 \end{pmatrix}. \tag{5}$$

Here $\boldsymbol{\tau}$ denotes the Pauli matrices acting on the subspace $\mathbb{C}^2$ of $\mathbb{C}^3$. Since $U(-\boldsymbol{n}) \neq U(\boldsymbol{n})$, the model or state obtained by the unitary transformation $U(\boldsymbol{n})$ is in general a model or state on S^2, not on RP^2. However,

$$U(-\boldsymbol{n}) = U(\boldsymbol{n}) Z, \quad Z = \begin{pmatrix} -\mathbf{1} & \\ & 1 \end{pmatrix}, \tag{6}$$

and hence, if the model or state has the Z symmetry, then a model on RP^2 is obtained by using $U(\boldsymbol{n})$.

Take the Hamiltonian whose ground-state manifold is the set of DDW states to be

$$H_{\rm DDW}(\boldsymbol{n}) = -\sum_j \sigma_j^z(-|u(\boldsymbol{n})\rangle\langle u(\boldsymbol{n})| + |w(\boldsymbol{n})\rangle\langle w(\boldsymbol{n})|)_{j+1/2}\sigma_{j+1}^z \tag{7}$$

$$= -\sum_j \sigma_j^z \begin{pmatrix} n_z^2 & (n_x - in_y)n_z & 0 \\ (n_x + in_y)n_z & n_x^2 + n_y^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{j+1/2} \sigma_{j+1}^z. \tag{8}$$

It satisfies $H_{\rm DDW}(-\boldsymbol{n}) = H_{\rm DDW}(\boldsymbol{n})$, so it is a model on RP^2. There is no need to use the state $|v(\boldsymbol{n})\rangle$. Using the unitary transformation, $H_{\rm DDW}(\boldsymbol{n})$ can also be written as

$$H_{\rm DDW}(\boldsymbol{n}) = U(\boldsymbol{n})H_{\rm DDW}^0 U(\boldsymbol{n})^{-1}, \tag{9}$$

$$H_{\rm DDW}^0 = -\sum_j \sigma_j^z \begin{pmatrix} -1 & & \\ & 0 & \\ & & 1 \end{pmatrix}_{j+1/2} \sigma_{j+1}^z. \tag{10}$$

Note that $H_{\rm DDW}^0$ has the Z symmetry.

The term that mixes the DDW states is given by

$$H_{\rm fl}(\boldsymbol{n}) = U(\boldsymbol{n})H_{\rm fl}^0 U(\boldsymbol{n})^{-1}, \tag{11}$$

$$H_{\rm fl}^0 = -\sum_j \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix}_{j-1/2} \sigma_j^x \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix}_{j+1/2}. \tag{12}$$

Since

$$\begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix} Z = -Z \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix}, \tag{13}$$

the two factors of -1 cancel, and $H_{\rm fl}^0$ has the Z symmetry. Therefore $H_{\rm fl}(\boldsymbol{n})$ is a model on RP^2. Of course, this can also be checked directly:

$$H_{\rm fl}(\boldsymbol{n}) = -\sum_j \begin{pmatrix} 0 & 0 & n_z \\ 0 & 0 & n_x + in_y \\ n_z & n_x - in_y & 0 \end{pmatrix}_{j-1/2} \sigma_j^x \begin{pmatrix} 0 & 0 & n_z \\ 0 & 0 & n_x + in_y \\ n_z & n_x - in_y & 0 \end{pmatrix}_{j+1/2}. \tag{14}$$

Finally, take the parameter dependence in the S^1 direction of the adiabatic evolution to be the unitary transformation

$$V(t) = \prod_j e^{-\frac{it}{4}\sigma_j^x}. \tag{15}$$

Then

$$V(2\pi) \sim \prod_j \sigma_j^x \tag{16}$$

is the operator of the $\mathbb{Z}_2$ symmetry. Since the model has this $\mathbb{Z}_2$ symmetry, the Hamiltonian is periodic in $t \in [0, 2\pi]$.

Collecting the above, the model is

$$H(\boldsymbol{n},t) = V(t)U(\boldsymbol{n})H^0U(\boldsymbol{n})^{-1}V(t)^{-1}, \tag{17}$$

$$H^0 = -\sum_j \sigma_j^z \begin{pmatrix} -1 & & \\ & 0 & \\ & & 1 \end{pmatrix}_{j+1/2} \sigma_{j+1}^z - \sum_j \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix}_{j-1/2} \sigma_j^x \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix}_{j+1/2}, \tag{18}$$

$$U(\boldsymbol{n}) = \bigotimes_j \begin{pmatrix} n_z & n_x - in_y & 0 \\ n_x + in_y & -n_z & 0 \\ 0 & 0 & 1 \end{pmatrix}_{j+1/2}, \tag{19}$$

$$V(t) = \prod_j e^{-\frac{it}{4}\sigma_j^x}. \tag{20}$$

Comment on the $S^2 \times S^1$ model—

By the same method, one cannot obtain an $S^2 \times S^1$ model in which a wave function of Chern number 1 is pumped. The modification would be to write $\boldsymbol{n} = (\theta, \phi)$ in spherical coordinates and define

$$U'(\theta,\phi) := (|u'(\boldsymbol{n})\rangle, |v'(\boldsymbol{n})\rangle, |w'(\boldsymbol{n})\rangle) = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\phi} & 0 \\ \sin\frac{\theta}{2}e^{i\phi} & -\cos\frac{\theta}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{21}$$

In this gauge, however, the south pole $\theta = \pi$ is

$$U'(\theta = \pi, \phi) := \begin{pmatrix} 0 & e^{-i\phi} & 0 \\ e^{i\phi} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{22}$$

and the ϕ dependence remains. On the other hand, the Hamiltonian H^0 does not commute with this unitary transformation $U'(\theta = \pi, \phi)$, and hence one cannot construct a model on S^2 using the unitary transformation $U'(\theta, \phi)$.

Also, the Hamiltonian $H(\boldsymbol{n}, t)$ on $RP^2 \times S^1$ constructed above can be regarded as a model on $S^2 \times S^1$, but the Chern number of the state $|u(\boldsymbol{n})\rangle$ on S^2 is zero.

1.1 MPS

Start from the MPS of the cluster model,

$$H = -\sum_j \sigma_{j-1}^z \sigma_j^x \sigma_{j+1}^z. \tag{23}$$

The MPS is

$$A^\uparrow = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A^\downarrow = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}. \tag{24}$$

By adding degrees of freedom to this MPS and applying the unitary transformation, one obtains the MPS of the $RP^2 \times S^1$ model.

2 The $L(3,1) \times S^1$ Model

The $L(3,1) \times S^1$ model can be constructed in a similar way. The lens space $L(3,1)$ is given by

$$\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}/(z_1, z_2) \sim (\omega z_1, \omega z_2), \quad \omega = e^{2\pi i/3}. \tag{25}$$

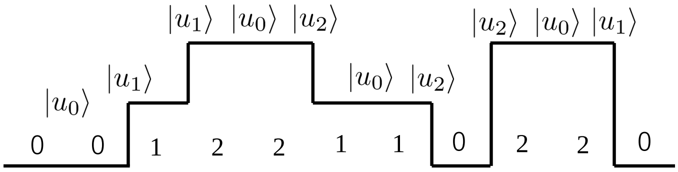

Figure 1

The pair (z_1, z_2) can be regarded as a point on S^3. Introduce the three basis states parametrized by (z_1, z_2) as

$$U = (|u_0\rangle, |u_1\rangle, |u_2\rangle) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z_1 & z_2^* \\ 0 & z_2 & -z_1^* \end{pmatrix}. \tag{26}$$

Under the $\mathbb{Z}_3^\tau$ transformation, this behaves as

$$U \mapsto (|u_0\rangle, \omega |u_1\rangle, \omega^2 |u_2\rangle) = UZ_\tau, \quad Z_\tau = \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix}. \tag{27}$$

Prepare on each bond the Hilbert space spanned by $|u_0\rangle, |u_1\rangle, |u_2\rangle$.

Place on each site the group-algebra degree of freedom of $\mathbb{Z}_3^\sigma$, and write $\sigma_j \in \{0, 1, 2\}$. Define

$$\Delta\sigma_j = \sigma_{j+1} - \sigma_j \mod 3. \tag{28}$$

For a state

$$|\sigma_1\sigma_2\cdots\rangle, \tag{29}$$

construct the DDW state by assigning the bond state according to Δ_j as follows:

- if $\Delta\sigma_j = 0$, put $|u_0\rangle$;
- if $\Delta\sigma_j = 1$, put $|u_1\rangle$;
- if $\Delta\sigma_j = 2$, put $|u_2\rangle$.

See the figure. The point is that, under PBC, the sum of the differences is fixed:

$$\sum_j \Delta\sigma_j = 0 \mod 3. \tag{30}$$

Therefore, under the $\mathbb{Z}_3^\tau$ transformation on the bonds,

$$|\mathrm{DDW}\rangle \mapsto |\mathrm{DDW}\rangle \prod_j \omega^{\Delta\sigma_j} = |\mathrm{DDW}\rangle\, \omega^{\sum_j \Delta\sigma_j} = |\mathrm{DDW}\rangle, \tag{31}$$

so the DDW state is $\mathbb{Z}_3^\tau$ invariant. Consequently, the equal-weight superposition of DDW states,

$$|\mathrm{SG}\rangle = \sum_{\mathrm{DDW}} |\mathrm{DDW}\rangle, \tag{32}$$

is $\mathbb{Z}_3^\tau$ invariant. It is also invariant under the $\mathbb{Z}_3$ transformation that permutes the spins on the sites,

$$X_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{33}$$

Thus it is $\mathbb{Z}_3^\sigma \times \mathbb{Z}_3^\tau$ invariant. This model is probably a model of an SPT phase with $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry.

For the S^1 dependence, one should construct it so that, at $t = 2\pi$, it becomes the transformation X_σ that permutes the $\mathbb{Z}_3$ spins on the sites. Thus introduce the t-dependent unitary transformation

$$V(t) = W \begin{pmatrix} 1 & & \\ & e^{\frac{it}{3}} & \\ & & e^{\frac{2it}{3}} \end{pmatrix} W^\dagger, \quad W = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \\ 1 & 1 & 1 \end{pmatrix}. \tag{34}$$

Then

$$V(2\pi) = X_\sigma. \tag{35}$$

References

[1] Ken Shiozaki, Masatoshi Sato, Kiyonori Gomi, *Topological Crystalline Materials - General Formulation, Module Structure, and Wallpaper Groups* -, arXiv:1701.08725.