A Note on Redefining the Second Stiefel–Whitney Class

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The Whitney sum formula

$$w(E+F) = w(E)w(F) \tag{1}$$

implies

$$w_1(E+F) = w_1(E) + w_1(F), \tag{2}$$
$$w_2(E+F) = w_2(E) + w_2(F) + w_1(E)w_1(F). \tag{3}$$

We would like to ask whether w_2 can be redefined so that

$$w_2'(E+F) = w_2'(E) + w_2'(F) \tag{4}$$

holds. This is similar to a quadratic refinement.

In general, for a symmetric bilinear form $b(x,y)$ with values in $\mathbb{Z}_2$, there exists a $\mathbb{Z}_2$-valued function $q(x)$ satisfying

$$q(x+y) = q(x) + q(y) + b(x,y). \tag{5}$$

Such a q is called a quadratic refinement. It is not unique. In the present situation,

$$b(E,F) := w_1(E)w_1(F) \tag{6}$$

is a symmetric bilinear form. Thus there exists a quadratic refinement $q(E)$. Using it, define

$$w_2'(E) := w_2(E) + q(E). \tag{7}$$

Then

$$\begin{aligned} w_2'(E+F) &= w_2(E+F) + q(E+F) && (8)\\ &= w_2(E) + w_2(F) + w_1(E)w_1(F) + q(E) + q(F) + w_1(E)w_1(F) && (9)\\ &= w_2'(E) + w_2'(F). && (10) \end{aligned}$$

How should one choose $q(E)$? As a general principle, after restricting the base space on which the principal bundle is defined, a quadratic refinement should be defined in terms of topological invariants on lower-dimensional base spaces [1]. Let us explicitly construct it when the base space is the two-dimensional torus T^2.

Write

$$w_1(E) = \nu_{1x}(E)dx + \nu_{1y}(E)dy, \tag{11}$$

and

$$w_2(E) = \nu_2(E)dxdy. \tag{12}$$

Then the sum rule is

$$\nu_2(E+F) = \nu_2(E) + \nu_2(F) + \nu_{1x}(E)\nu_{1y}(F) + \nu_{1y}(E)\nu_{1x}(F). \tag{13}$$

For the symmetric bilinear form

$$b(E,F)=\nu_{1x}(E)\nu_{1y}(F)+\nu_{1y}(E)\nu_{1x}(F), \tag{14}$$

one can take the quadratic refinement

$$q(E)=\nu_{1x}(E)\nu_{1y}(E). \tag{15}$$

Indeed,

$$\begin{aligned} q(E+F) &= (\nu_{1x}(E)+\nu_{1x}(F))(\nu_{1y}(E)+\nu_{1y}(F)) && (16)\\ &= \nu_{1x}(E)\nu_{1y}(E)+\nu_{1x}(F)\nu_{1y}(F)+\nu_{1x}(E)\nu_{1y}(F)+\nu_{1x}(F)\nu_{1y}(E) && (17)\\ &= q(E)+q(F)+b(E,F). && (18) \end{aligned}$$

Therefore, redefining

$$\widetilde{\nu_2}(E):=\nu_2(E)-\nu_{1x}(E)\nu_{1y}(E), \tag{19}$$

we obtain the additive rule

$$\widetilde{\nu_2}(E+F)=\widetilde{\nu_2}(E)+\widetilde{\nu_2}(F). \tag{20}$$

Such a redefinition is also discussed in [2].

To see the viewpoint based on topological invariants of subspaces, it is useful to write the construction more formally. Let S^1_x and S^1_y be the subcycles of T^2 in the k_x and k_y directions, respectively. The restriction maps to these circles define homomorphisms

$$r_\mu : H^*(S^1_\mu;\mathbb{Z}/2)\to H^*(T^2;\mathbb{Z}/2). \tag{21}$$

The restriction of E to S^1_μ determines an element

$$r_\mu w_1(E|_{S^1_\mu})\in H^1(T^2;\mathbb{Z}/2). \tag{22}$$

The quadratic refinement is then defined by

$$q(E)=r_x w_1(E|_{S^1_x})\cup r_y w_1(E|_{S^1_y}). \tag{23}$$

References

[1] Seishiro Ono and Ken Shiozaki, *Towards complete characterization of topological insulators and superconductors: A systematic construction of topological invariants based on Atiyah-Hirzebruch spectral sequence*, arXiv:2311.15814.

[2] Yanzhu Chen, Sheng-Jie Huang, Yi-Ting Hsu, and Tzu-Chieh Wei, *Topological invariants beyond symmetry indicators: Boundary diagnostics for twofold rotationally symmetric superconductors*, arXiv:2109.06959.