Axioms for Spectral Sequences (Draft)

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1 Axioms

Following Chapter XV, Section 7 of [1], I summarize the axioms for spectral sequences and their consequences.

Assume that a module $H(p,q)$ is assigned to each pair $p,q \in \mathbb{Z}, -\infty \leq p \leq q \leq \infty$.

$$H(p) = H(p,\infty), \quad H = H(-\infty) = H(-\infty,\infty) \tag{1.1}$$

and so on. Introduce the notation

$$(p,q) \leq (p',q') \Leftrightarrow p \leq p', q \leq q'. \tag{1.2}$$

The notation $(p,q,r) \leq (p',q',r')$ is used similarly.

The following are the axioms, as in Chapter XV, Section 7 of [1].

Assume that the following homomorphisms are defined:

$$H(p',q') \to H(p,q), \quad (p,q) \leq (p',q'), \tag{1.3}$$

$$H(p,q) \xrightarrow{\delta} H(q,r), \quad -\infty \leq p \leq q \leq r \leq \infty. \tag{1.4}$$

(SP.1) The following map is an isomorphism:

$$H(p,q) \xrightarrow{\cong} H(p,q). \tag{1.5}$$

(SP.2) For $(p,q) \leq (p',q') \leq (p'',q'')$, the following diagram is commutative:

$$\begin{tikzcd}
H(p'',q'') \arrow[r] \arrow[dr] & H(p,q) \\
& H(p',q') \arrow[u]
\end{tikzcd} \tag{1.6}$$

(SP.3) For $(p,q,r) \leq (p',q',r')$, the following diagram is commutative:

$$\begin{tikzcd}
H(p',q') \arrow[r, "\delta"] \arrow[d] & H(q',r') \arrow[d] \\
H(p,q) \arrow[r, "\delta"] & H(q,r)
\end{tikzcd} \tag{1.7}$$

(SP.4) For $p \leq q \leq r$, the following sequence is exact:

$$\cdots \to H(q,r) \to H(p,r) \to H(p,q) \xrightarrow{\delta} H(q,r) \to \cdots . \tag{1.8}$$

(SP.5) Fix q. The direct system

$$H(q,q) \to H(q-1,q) \to \cdots \to H(p,q) \to H(p-1,q) \to \cdots \to H(-\infty,q) \tag{1.9}$$

has $H(-\infty,q)$ as its direct limit.

I do not introduce a special symbol for the first homomorphism $H(p',q') \to H(p,q)$. When $H(p,q)$ is graded, the first homomorphism is assumed to have degree 0, while δ is assumed to have degree 1.

As a remark, in the graded case of (SP.4), one should note that the degree changes by 1 after going once around the sequence, because δ changes the degree. Also, (SP.5) is not an inclusion map. By (SP.1) and (SP.4),

$$\to H(p,p) \xrightarrow{\cong} H(p,p) \xrightarrow{\cong} H(p,p) \xrightarrow{\delta} H(p,p) \to \tag{1.10}$$

is an exact sequence. Therefore, unless

$$H(p,p) = 0, \tag{1.11}$$

one obtains a contradiction.

2 Weak Axioms

In [1], it is claimed that the axioms above follow even if, in the definition of δ, one only defines the case $r = \infty$:

$$\delta : H(p,q) \to H(q), \quad p \leq q. \tag{2.1}$$

I examine this point.

Consider the following weak axioms.

Assume that the following homomorphisms are defined:

$$H(p',q') \to H(p,q), \quad (p,q) \leq (p',q'), \tag{2.2}$$

$$H(p,q) \xrightarrow{\delta} H(q), \quad -\infty \leq p \leq q \leq \infty. \tag{2.3}$$

(SP.1) The following map is an isomorphism:

$$H(p,q) \xrightarrow{\cong} H(p,q). \tag{2.4}$$

(SP.2) For $(p,q) \leq (p',q') \leq (p'',q'')$, the following diagram is commutative:

$$\begin{tikzcd} H(p'',q'') \arrow[rr] \arrow[dr] & & H(p,q) \\ & H(p',q') \arrow[ur] & \end{tikzcd} \tag{2.5}$$

(SP.3') For $(p,q) \leq (p',q')$, the following diagram is commutative:

$$\begin{tikzcd}
H(p',q') \arrow[r, "\delta"] \arrow[d] & H(q') \arrow[d] \\
H(p,q) \arrow[r, "\delta"] & H(q)
\end{tikzcd} \tag{2.6}$$

(SP.4') For $p \leq q$, the following sequence is exact:

$$\cdots \to H(q) \to H(p) \to H(p,q) \xrightarrow{\delta} H(q) \to \cdots . \tag{2.7}$$

(SP.5) Fix q. The direct system

$$H(q,q) \to H(q-1,q) \to \cdots \to H(p,q) \to H(p-1,q) \to \cdots \to H(-\infty,q) \tag{2.8}$$

has $H(-\infty,q)$ as its direct limit.

Note that the connecting homomorphism

$$H(p,q) \xrightarrow{\delta} H(q) \to H(q,r), \quad p \leq q \leq r, \tag{2.9}$$

defines $\delta' : H(p,q) \xrightarrow{\delta} H(q,r)$. The claim in [1] is that the weak axioms above imply the axioms in the preceding section.

By (SP.2) and (SP.3'), extending the commutative diagram to the right gives

$$\begin{tikzcd}
H(p',q') \arrow[r, "\delta"] \arrow[d] & H(q') \arrow[r] \arrow[d] \arrow[dr] & H(q',r') \arrow[d] \\
H(p,q) \arrow[r, "\delta"] & H(q) \arrow[r] & H(q,r)
\end{tikzcd} \tag{2.10}$$

and hence one obtains (SP.3).

Next I derive (SP.4). I prove that the following sequence is exact:

$$\longrightarrow H(q,r) \xrightarrow{i'} H(p,r) \xrightarrow{j'} H(p,q) \xrightarrow{\delta'} H(q,r) \longrightarrow \tag{2.11}$$

First, I show $\ker \delta' = \operatorname{im} j'$. The following diagram is commutative, and the maps with common subscripts i, j, δ form exact sequences:

$$\begin{tikzcd}
H(p) & & \\
H(r) \arrow[u, "i_{pr}"] \arrow[r, "i_{qr}"] & H(q) \arrow[ul, "i_{pq}"'] \arrow[dr, "j_{qr}"] & \\
H(p,r) \arrow[u, "\delta_{pr}"] \arrow[r, "j'"] & H(p,q) \arrow[u, "\delta_{pq}"] \arrow[r, "\delta'"] & H(q,r) \\
H(p) \arrow[u, "j_{pr}"] \arrow[ur, "j_{pq}"] & &
\end{tikzcd} \tag{2.12}$$

From the diagram,

$$\delta' j' = j_{qr} \delta_{pq} j' = j_{qr} i_{qr} \delta_{pr} = 0. \tag{2.13}$$

Suppose $\delta'(x) = 0$. Since $j_{qr}\delta_{pq}(x) = 0$, there exists $y \in H(r)$ such that $\delta_{pq}(x) = i_{qr}(y)$. Then $i_{pr}(y) = i_{pq}i_{qr}(y) = i_{pq}\delta_{pq}(y) = 0$, so there exists $z \in H(p,r)$ with $y = \delta_{pr}(z)$. Therefore

$$\delta_{pq}(x) = i_{qr}(y) = i_{qr}\delta_{pr}(z) = \delta_{pq}j'(z). \tag{2.14}$$

It follows that $x - j'(z) \in \ker \delta_{pq} = \operatorname{im} j_{pq} = \operatorname{im} j' j_{pr}$, and hence $x \in \operatorname{im} j'$.

Next I show $\ker i' = \operatorname{im} \delta'$. The following diagram is commutative, and the maps with common subscripts i, j, δ form exact sequences:

$$\begin{tikzcd}
 & H(r) \arrow[d, "i_{qr}"] \arrow[dr, "i_{pr}"] & \\
 & H(q) \arrow[r, "i_{pq}"] \arrow[d, "j_{qr}"] & H(p) \arrow[d, "j_{pr}"] \\
H(p,q) \arrow[r, "\delta'"] \arrow[ur, "\delta_{pq}"] & H(q,r) \arrow[r, "i'"] \arrow[d, "\delta_{qr}"] & H(p,r) \arrow[dl, "\delta_{pr}"] \\
 & H(r) &
\end{tikzcd} \tag{2.15}$$

From the diagram,

$$i'\delta' = i'j_{qr}\delta_{pq} = j_{pr}i_{pq}\delta_{pq} = 0. \tag{2.16}$$

Suppose $i'(x) = 0$. Since $\delta_{pr}i'(x) = \delta_{qr}(x) = 0$, we can write $x = j_{qr}(y)$ with $y \in H(q)$. Then $0 = i'(x) = i'j_{qr}(y) = j_{pr}i_{pq}(y)$, so we can write $i_{pq}(y) = i_{pr}(z) = i_{pq}i_{qr}(z)$ for some $z \in H(r)$. Therefore $y - i_{qr}(z) \in \ker i_{pq} = \operatorname{im} \delta_{pq}$, and hence $y = i_{qr}(z) + \delta_{pq}(w)$ for some $w \in H(p,q)$. Thus

$$x = j_{qr}(y) = j_{qr}\delta_{pq}(w) = \delta'(w). \tag{2.17}$$

Finally I show $\ker j' = \operatorname{im} i'$. First, $j'i' = 0$ follows from the commutative diagram

$$\begin{tikzcd}
H(q,r) \arrow[r, "i'"] \arrow[dr] & H(p,r) \arrow[r, "j'"] & H(p,q) \\
 & H(q,q) = 0 \arrow[ur] &
\end{tikzcd} \tag{2.18}$$

To prove $\ker j' \subset \operatorname{im} i'$, use the following commutative diagram:

$$\begin{tikzcd}
H(q) \arrow[r, "i_{pq}"] \arrow[d, "i_{qr}"] & H(p) \arrow[d, "j_{pr}"] \arrow[dr, "j_{pq}"] & \\
H(q,r) \arrow[r, "i'"] \arrow[dr, "\delta_{qr}"] & H(p,r) \arrow[r, "j'"] \arrow[d, "\delta_{pr}"] & H(p,q) \arrow[d, "\delta_{pq}"] \\
 & H(r) \arrow[r, "i_{qr}"] & H(q)
\end{tikzcd} \tag{2.19}$$

Suppose $j'(x) = 0$. Since $\delta_{pq}j'(x) = i_{qr}\delta_{pr}(x) = 0$, we can write $\delta_{pr}(x) = \delta_{qr}(y) = \delta_{pr}i'(y)$ for some $y \in H(q,r)$. Then $x - i'(y) \in \ker \delta_{pr} = \operatorname{im} j_{pr}$, so $x = i'(y) + j_{pr}(z)$ for some $z \in H(p)$. Since $0 = j'(x) = j'j_{pr}(z) = j_{pq}(z)$, we can write $z = i_{pq}(w)$ for some $w \in H(q)$. Therefore

$$x = i'(y) + j_{pr}i_{pq}(w) = i'(y) + i'i_{qr}(w) \in \operatorname{im} i'. \tag{2.20}$$

This proves (SP.4). □

Although it appeared in the proof above, note that, for $p \le q \le r$, the homomorphism

$$H(q,r) \to H(p,q) \tag{2.21}$$

always factors through $H(q,q) = 0$, and is therefore always the zero map.

2.1 Example: Generalized Homology Theories

As an example, with the classification of invertible phases in mind, I check whether the axioms above are satisfied for the Atiyah-Hirzebruch spectral sequence of a generalized homology theory. Let

$$X_0 \subset X_1 \subset \cdots \subset X_d = X \tag{2.22}$$

be a filtration of a space X. Set

$$H(p,q) = \bigoplus_{n\in\mathbb{Z}} h_n(X_{-p}, X_{-q}), \quad p \le q. \tag{2.23}$$

In what follows, the notation (p,q) means $p \ge q$.

- The map $h_n(X_{p'}, X_{q'}) \to h_n(X_p, X_q), (p,q) \ge (p', q')$, is the homomorphism induced by inclusion.
- The map $\delta : h_n(X_p, X_q) \to h_{n-1}(X_q)$ is the boundary homomorphism.
- For $(p'', q'') \le (p', q') \le (p,q)$, the commutativity of the diagram

  $$\begin{tikzcd} h_n(X_{p''}, X_{q''}) \arrow[rr] \arrow[dr] & & h_n(X_p, X_q) \\ & h_n(X_{p'}, X_{q'}) \arrow[ur] & \end{tikzcd} \tag{2.24}$$

  follows from the composition of inclusions.
- For $(p', q') \le (p,q)$, the commutativity of the diagram

  $$\begin{tikzcd} h_n(X_{p'}, X_{q'}) \arrow[r, "\delta"] \arrow[d] & h_{n-1}(X_{q'}) \arrow[d] \\ h_n(X_p, X_q) \arrow[r, "\delta"] & h_{n-1}(X_q) \end{tikzcd} \tag{2.25}$$

  also follows from the compatibility of the boundary operator with inclusion maps.
- For $p \ge q$, the sequence

  $$\cdots \to h_n(X_q) \to h_n(X_p) \to h_n(X_p, X_q) \overset{\delta}{\to} h_{n-1}(X_q) \to \cdots \tag{2.26}$$

  is the long exact sequence of a pair.
- In the direct system

  $$0 = h_n(X_q, X_q) \to h_n(X_{q-1}, X_q) \to \cdots \to h_n(X_p, X_q) \to h_n(X_{p-1}, X_q) \to \cdots \to h_n(X, X_q), \tag{2.27}$$

  the existence of the limit $h_n(X, X_q)$ is guaranteed if X is finite-dimensional.

3 Consequences of the Axioms

First, I check the following statement.

Lemma 3.1 (Lemma 1.1 in [1])**.** *In the following commutative diagram, assume that the row is exact:*

$$\begin{tikzcd} & C \arrow[d, "\phi"] \arrow[dr, "\psi"] & \\ A' \arrow[ur] \arrow[r, "\phi'"] & A \arrow[r, "\eta"] & A'' \end{tikzcd} \tag{3.1}$$

Then η gives the following isomorphism:

$$\operatorname{im}\phi/\operatorname{im}\phi' \cong \operatorname{im}\psi. \tag{3.2}$$

Proof. By the commutativity of the diagram, note that $\operatorname{im}\phi' \subset \operatorname{im}\phi$. We have $\operatorname{im}\phi/\operatorname{im}\phi' = \operatorname{im}\phi/\ker\eta$. For $\phi(c) + \ker\eta \in \operatorname{im}\phi/\ker\eta$,

$$\eta(\phi(c) + \ker\eta) = \eta(\phi(c)) = \psi(c). \tag{3.3}$$

For an element $c' \in \ker\phi$, one has $\psi(c') = 0$, and hence this defines a well-defined map $\eta : \operatorname{im}\phi/\ker\eta \to \operatorname{im}\psi$.

Conversely, for $y \in \operatorname{im}\psi$, choose one $c \in C$ such that $y = \psi(c) = \eta(\phi(c))$. Then $\phi(c) \in \operatorname{im}\phi$ is determined up to $\ker\eta$. In other words, $\phi(c) + \ker\eta \in \operatorname{im}\phi/\ker\eta$ is determined. The ambiguity in the choice of c is harmless: if $\psi(c) = \eta(\phi(c)) = 0$, then $\phi(c) \in \ker\eta = 0$ in $\operatorname{im}\phi/\ker\eta$. □

Introduce the following notation:

$$\begin{aligned} F^pH &= \operatorname{im}[H(p) \to H], && (3.4)\\ Z_r^p &= \operatorname{im}[H(p, p+r) \to H(p, p+1)], \quad r \geq 1, && (3.5)\\ B_r^p &= \operatorname{im}[H(p-r+1, p) \xrightarrow{\delta} H(p, p+1)], \quad r \geq 1, && (3.6)\\ E_r^p &= Z_r^p/B_r^p, \quad r \geq 1. && (3.7) \end{aligned}$$

1) There is the following filtration of H:

$$\cdots \subset F^{p+1}H \subset F^pH \subset \cdots \subset F^{-\infty}H = H. \tag{3.8}$$

Proof. This follows from the commutativity of the diagram

$$\begin{tikzcd} H(p+1) \arrow[rr] \arrow[dr] & & H \\ & H(p) \arrow[ur] & \end{tikzcd} \tag{3.9}$$

2) There are the following inclusions:

$$\cdots B_r^p \subset B_{r+1}^p \subset \cdots B_\infty^p \subset Z_\infty^p \subset \cdots \subset Z_{r+1}^p \subset Z_r^p \subset \cdots \tag{3.10}$$

Proof. The inclusion $B_r^p \subset B_{r+1}^p$ follows from the commutativity of the diagram

$$\begin{tikzcd} H(p-r+1, p) \arrow[r, "\delta"] \arrow[d] & H(p, p+1) \arrow[d, "\cong"] \\ H(p-r, p) \arrow[r, "\delta"] & H(p, p+1) \end{tikzcd} \tag{3.11}$$

The inclusion $Z^p_{r+1} \subset Z^p_r$ follows from the commutativity of the diagram

$$\begin{tikzcd}
H(p,p+r+1) \arrow[r] \arrow[d] & H(p,p+1) \arrow[d, "\cong"] \\
H(p,p+r) \arrow[r] & H(p,p+1)
\end{tikzcd} \tag{3.12}$$

The inclusion $B^p_\infty \subset Z^p_\infty$ follows from the commutativity of the diagram

$$\begin{tikzcd}
H(-\infty,p) \arrow[r, "\delta"] \arrow[d, "\cong"] & H(p,\infty) \arrow[d] \\
H(-\infty,p) \arrow[r, "\delta"] & H(p,p+1)
\end{tikzcd} \tag{3.13}$$

3) There is the following isomorphism:

$$\delta^p_r : Z^p_r/Z^p_{r+1} \cong B^{p+r}_{r+1}/B^{p+r}_r. \tag{3.14}$$

Proof. In the following commutative diagram, the row is exact:

$$\begin{tikzcd}
& H(p,p+r) \arrow[d] \arrow[dr] & \\
H(p,p+r+1) \arrow[ur] \arrow[r] & H(p,p+1) \arrow[r, "\delta"] & H(p+1,p+r+1)
\end{tikzcd} \tag{3.15}$$

Here the upper-right map is the following connecting homomorphism:

$$H(p,p+r) \xrightarrow{\delta} H(p+r,p+r+1) \to H(p+1,p+r+1). \tag{3.16}$$

By the lemma above,

$$\operatorname{im}[H(p,p+r) \to H(p,p+1)]/\operatorname{im}[H(p,p+r+1) \to H(p,p+1)] \tag{3.17}$$
$$\cong \operatorname{im}[H(p,p+r) \to H(p+1,p+r+1)]. \tag{3.18}$$

This is

$$Z^p_r/Z^p_{r+1} \cong \operatorname{im}[H(p,p+r) \to H(p+1,p+r+1)]. \tag{3.19}$$

Also, in the following commutative diagram, the row is exact:

$$\begin{tikzcd}
& H(p,p+r) \arrow[d, "\delta"] \arrow[dr] & \\
H(p+1,p+r) \arrow[ur] \arrow[r, "\delta"] & H(p+r,p+r+1) \arrow[r] & H(p+1,p+r+1)
\end{tikzcd} \tag{3.20}$$

Again by the lemma above,

$$\operatorname{im}[H(p,p+r) \xrightarrow{\delta} H(p+r,p+r+1)]/\operatorname{im}[H(p+1,p+r) \xrightarrow{\delta} H(p+r,p+r+1)] \tag{3.21}$$
$$\cong \operatorname{im}[H(p,p+r) \to H(p+1,p+r+1)]. \tag{3.22}$$

This is

$$B^{p+r+1}_r/B^{p+r}_{r+1} \cong \operatorname{im}[H(p,p+r) \to H(p+1,p+r+1)]. \tag{3.23}$$

Therefore

$$Z_r^p/Z_{r+1}^p \cong \mathrm{im}\,[H(p,p+r) \to H(p+1,p+r+1)] \cong B_r^{p+r+1}/B_{r+1}^{p+r}. \quad \square \tag{3.24}$$

Write this isomorphism as

$$\delta_r^p : Z_r^p/Z_{r+1}^p \cong B_r^{p+r+1}/B_{r+1}^{p+r}. \tag{3.25}$$

Define the following composite homomorphism

$$E_r^p = Z_r^p/B_r^p \twoheadrightarrow Z_r^p/Z_{r+1}^p \xrightarrow[\cong]{\delta_r^p} B_{r+1}^{p+r}/B_r^{p+r} \hookrightarrow Z_r^{p+r}/B_r^{p+r} = E_r^{p+r} \tag{3.26}$$

to be

$$d_r^p : E_r^p \to E_r^{p+r}. \tag{3.27}$$

Then one obtains

$$\ker d_r^p = \ker[Z_r^p/B_r^p \to Z_r^p/Z_{r+1}^p] \cong Z_{r+1}^p/B_r^p, \tag{3.28}$$

$$\mathrm{im}\, d_r^p = \mathrm{im}\,[B_{r+1}^{p+r}/B_r^{p+r} \hookrightarrow Z_r^{p+r}/B_r^{p+r}] \cong B_{r+1}^{p+r}/B_r^{p+r}. \tag{3.29}$$

From this, for the sequence

$$E_r^{p-r} \xrightarrow{d_r^{p-r}} E_r^p \xrightarrow{d_r^p} E_r^{p+r}, \tag{3.30}$$

one obtains

$$\mathrm{im}\, d_r^{p-r} = B_{r+1}^p/B_r^p \subset Z_{r+1}^p/B_r^p = \ker d_r^p. \tag{3.31}$$

This means that d_r is a differential. Moreover,

$$\ker d_r^p/\mathrm{im}\, d_r^{p-r} \cong Z_{r+1}^p/B_{r+1}^p = E_{r+1}^p \tag{3.32}$$

holds. If

$$E_r = \sum_p E_r^p, \tag{3.33}$$

we obtain the following theorem.

Theorem 3.2. *For $r \geq 1$, $d_r : E_r \to E_r$ is a differential of degree r. The homology of d_r is E_{r+1}.*

In particular, for $r = 1$, since $B_1^p = \mathrm{im}\,[0 = H(p,p) \xrightarrow{\delta} H(p,p+1)] = 0$, one has

$$E_1^p = Z_1^p = H(p,p+1). \tag{3.34}$$

Thus

$$d_1^p : H(p,p+1) \xrightarrow{\delta} H(p+1,p+2). \tag{3.35}$$

Consider another description of d_r^p. For $r \geq 1$, the following diagram is commutative and the row is exact:

$$\begin{tikzcd}
& H(p,p+r) \arrow[d] \arrow[dr] & \\
H(p-r+1,p) \arrow[ur, "\delta"] \arrow[r, "\delta"] & H(p,p+1) \arrow[r] & H(p-r+1,p+1)
\end{tikzcd} \tag{3.36}$$

Therefore, by Lemma 3.1,

$$\mathrm{im}\,[H(p,p+r)\to H(p,p+1)]/\mathrm{im}\,[H(p-r+1,p)\xrightarrow{\delta}H(p,p+1)]\cong\mathrm{im}\,[H(p,p+r)\to H(p-r+1,p+1)]. \tag{3.37}$$

That is,

$$E^p_r=Z^p_r/B^p_r\cong\mathrm{im}\,[H(p,p+r)\to H(p-r+1,p+1)]. \tag{3.38}$$

From the commutative diagram for $r\geq 1$,

$$\begin{tikzcd} H(p,p+r) \arrow[r, "\varphi^p_r"] \arrow[d, "\delta"] & H(p-r+1,p+1) \arrow[d, "\delta"] \\ H(p+r,p+2r) \arrow[r, "\varphi^{p+r}_r"] & H(p+1,p+r+1) \end{tikzcd} \tag{3.39}$$

one can define the homomorphism

$$\tilde{d}^p_r : E^p_r=\mathrm{im}\,\varphi^p_r\to\mathrm{im}\,\varphi^{p+r}_r=E^{p+r}_r, \tag{3.40}$$

$$\mathrm{im}\,\varphi^p_r\ni y=\varphi^p_r(x)\xrightarrow{\delta}\delta\varphi^p_r(x)=\varphi^{p+r}_r\delta(x)\in\mathrm{im}\,\varphi^{p+r}_r. \tag{3.41}$$

In particular, for $r=1$ this is

$$\tilde{d}^p_1 : H(p,p+1)\xrightarrow{\delta}H(p+1,p+2). \tag{3.42}$$

I leave open the verification that $\tilde{d}^p_r$ agrees with the previously defined d^p_r.

3.1 Example: Generalized Homology Theories

Take the following:

$$F_ph_n=\mathrm{im}\,[h_n(X_p)\to h_n(X)], \tag{3.43}$$

$$Z^r_{p,q}=\mathrm{im}\,[h_{p+q}(X_p,X_{p-r})\to h_{p+q}(X_p,X_{p-1})], \tag{3.44}$$

$$B^r_{p,q}=\mathrm{im}\,[h_{p+q+1}(X_{p+r-1},X_p)\xrightarrow{\delta}h_{p+q}(X_p,X_{p-1})], \tag{3.45}$$

$$E^r_{p,q}=Z^r_{p,q}/B^r_{p,q}. \tag{3.46}$$

- The meaning of $Z^r_{p,-p}$ is: “the classification obtained by taking invertible phases on the p-skeleton X_p, allowing anomalies on the $(p-r)$-skeleton X_{p-r}, and restricting them to the p-cells.”
- The meaning of $B^r_{p,-p}$ is: “the invertible phases on the p-cells that are obtained by adiabatic pumps on the $(p+r-1)$-skeleton X_{p+r-1}.”

These are the respective interpretations.

For $r=1$,

$$E^1_{p,q}=h_{p+q}(X_p,X_{p-1}), \tag{3.47}$$

$$d^1_{p,q} : h_{p+q}(X_p,X_{p-1})\to h_{p+q-1}(X_{p-1},X_{p-2}). \tag{3.48}$$

The differential $d^r_{p,q}:E^r_{p,q}\to E^r_{p-r,q+r-1}$ for $r>1$ can be understood as being defined by the following diagram:

$$d^r_{p,q} : \mathrm{im}\,\varphi^r_{p,q}\to\mathrm{im}\,\varphi^r_{p-r,q-1} \tag{3.49}$$

$$\begin{tikzcd}
h_{p+q}(X_p, X_{p-r}) \arrow[r, "\varphi^r_{p,q}"] \arrow[d, "\partial"] & h_{p+q}(X_{p+r-1}, X_{p-1}) \arrow[d, "\partial"] \\
h_{p+q-1}(X_{p-r}) \arrow[d] & h_{p+q-1}(X_{p-1}) \arrow[d] \\
h_{p+q-1}(X_{p-r}, X_{p-2r}) \arrow[r, "\varphi^r_{p-r,q-1}"] & h_{p+q-1}(X_{p-1}, X_{p-r-1})
\end{tikzcd} \tag{3.50}$$

In other words, roughly speaking, it may be regarded as the boundary operator

$$\partial : h_{p+q}(X_p, X_{p-r}) \to h_{p+q-1}(X_{p-r}). \tag{3.51}$$

References

[1] Cartan, Henri, and Samuel Eilenberg, *Homological algebra*, Vol. 28. Princeton university press, 1999.