On Local Formulae for Stiefel–Whitney Classes

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Abstract

A memo on McLaughlin [1].

1 Preparation

Let $P \to M$ be a principal $SO(n)$-bundle, and let $\{U_i\}_i$ be a good cover of M. Write the transition functions as

$$g_{ij}(x) \in SO(n), \quad x \in U_{ij}. \tag{1}$$

The cocycle condition is

$$g_{ij}(x)g_{jk}(x) = g_{ik}(x), \quad x \in U_{ijk}. \tag{2}$$

Denote the projection from $SO(n)$ to the Stiefel manifold $SO(n)/SO(k-1)$ by

$$g \mapsto \bar{g}. \tag{3}$$

It carries the natural left action

$$g \cdot \bar{h} = \overline{gh}. \tag{4}$$

The k-th Stiefel–Whitney class takes values in

$$w_k(P) \in H^k(M, \mathbb{Z}_2). \tag{5}$$

We construct a Cech cocycle representing $w_k(P)$ from the transition functions [1]. Recall that

$$H_p(SO(n)/SO(k-1), \mathbb{Z}_2) = \begin{cases} 0 & p = 1, \dots, k-2, \\ \mathbb{Z}_2 & p = k-1. \end{cases} \tag{6}$$

2 Construction

For $x \in U_{ij}$ choose a path

$$\tau^1_{ij}(x) : \Delta^1 = [0,1] \to SO(n)/SO(k-1) \tag{7}$$

such that

$$\tau^1_{ij}(x)|_{t=0} = \bar{e}, \quad \tau^1_{ij}(x)|_{t=1} = \bar{g}_{ij}(x). \tag{8}$$

Assume that the maps

$$\tau^1_{ij} : \Delta^1 \times U_{ij} \to SO(n)/SO(k-1) \tag{9}$$

can be chosen continuously. We write the concatenation of two paths with matching endpoints as

$$\gamma_0 * \gamma_1, \quad \gamma_0|_{t=1} = \gamma_1|_{t=0}, \tag{10}$$

with the convention that the path runs from left to right. The path

$$V^1_{ijk}(x) := \tau^1_{ij}(x) * (g_{ij}(x) \cdot \tau^1_{jk}(x)) * (\tau^1_{ik}(x))^{-1}, \quad x \in U_{ijk}, \tag{11}$$

is a closed loop by the cocycle condition. Indeed, from left to right it goes as

$$\bar{e} \to \bar{g}_{ij}(x) \to \overline{g_{ij}(x)g_{jk}(x)} = \bar{g}_{ik}(x) \to \bar{e}. \tag{12}$$

2.1 $w_2(P)$

For $k = 2$, since

$$H_1(SO(n), \mathbb{Z}_2) = \mathbb{Z}_2, \tag{13}$$

$V^1_{ijk}(x)$ is either a nontrivial cycle or a trivial cycle. This nontrivial/trivial distinction does not depend on the choice of $x \in U_{ijk}$, so we obtain a Cech cocycle

$$z^2_{ijk} := \begin{cases} 0 & (V^1_{ijk} \text{ is a trivial cycle}), \\ 1 & (V^1_{ijk} \text{ is a nontrivial cycle}). \end{cases} \tag{14}$$

We set $w_2(P) = [z^2_{ijk}]$.

Since $H_1(SO(n), \mathbb{Z}_2) = \mathbb{Z}_2$, there is a $\mathbb{Z}_2$ ambiguity in the choice of the path $\tau^1_{ij}(x \in U_{ij}) : e \rightsquigarrow g_{ij}(x) \in SO(n)$. This corresponds to the choice of lift $SO(n) \to \mathrm{Spin}(n)$.

2.2 $w_3(P)$

For $k = 3$, the loop $V^1_{ijk}(x)$ is a trivial cycle, so there exists a singular 2-simplex $\tau^2_{ijk}(x)$ with boundary $V^1_{ijk}(x)$:

$$^\exists\tau^2_{ijk}(x), \quad \text{s.t.} \quad \partial\tau^2_{ijk}(x) = V^1_{ijk}(x). \tag{15}$$

Assume that the maps

$$\tau^2_{ijk} : \Delta^2 \times U_{ijk} \to SO(n)/SO(2) \tag{16}$$

can be chosen continuously. Then the following linear combination of singular 2-simplices defines a 2-cycle:

$$V^2_{ijkl}(x) = g_{ij}(x) \cdot \tau^2_{jkl}(x) + \tau^2_{ikl}(x) + \tau^2_{ijl}(x) + \tau^2_{ijk}(x), \quad x \in U_{ijkl}. \tag{17}$$

Since $H_2(SO(n)/SO(2), \mathbb{Z}_2) = \mathbb{Z}_2$, we obtain a Cech 3-cocycle

$$z^3_{ijkl} := \begin{cases} 0 & (V^2_{ijkl} \text{ is a trivial cycle}), \\ 1 & (V^2_{ijkl} \text{ is a nontrivial cycle}). \end{cases} \tag{18}$$

We set $w_3(P) = [z^3_{ijkl}]$.

For $k > 3$, namely for $w_4, w_5, \ldots$, the construction proceeds in the same way.

References

[1] D. A. McLaughlin, *Local formulae for Stiefel-Whitney classes*, Manuscripta Math. 89, 1–13 (1996).