On the Degree of K-Groups, Symmetry Groups, and Factor Systems

Ken Shiozaki

June 1, 2026

We summarize the relation between the degree of a K-group and the symmetry data. The E_1 page of the AHSS was

$$E_1^{p,-n} = K^{p-n}(X_p, X_{p-1}) = \prod K^{p-n}(D^p, \partial D^p) = \prod K^{-n}(pt). \tag{1}$$

We want to know the symmetry at degree $-n$. Let the symmetry at degree 0 be

$$\hat{g} H \hat{g}^{-1} = (-1)^{c_g} H, \quad \hat{g}\hat{h} = z_{g,h}\widehat{gh}, \quad \hat{g} = u_g K^{\phi_g}, \quad g \in G. \tag{2}$$

Shifting the degree to $-n$ is equivalent to adding the following chiral symmetries:

$$\gamma_i H \gamma_i^{-1} = -H, \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij}, \quad \hat{g}\gamma_i = (-1)^{c_g}\gamma_i \hat{g}. \tag{3}$$

First consider $n = 1$. Adding a chiral symmetry γ doubles the group elements. We write the new elements as $\{\gamma g\}_{g \in G}$:

$$G^{(n=1)} = G + \gamma G. \tag{4}$$

Here $G^{(n)}$ denotes the group at degree $-n$, so $G^{(0)} = G$. At degree $-n$, let $\phi_g^{(n)} \in \{0, 1\}$ specify whether $g \in G^{(n)}$ is unitary or antiunitary, and let $c_g^{(n)} \in \{0, 1\}$ specify whether it commutes or anticommutes with the Hamiltonian. Clearly,

$$\phi_{\gamma g}^{(1)} = 1, \quad c_{\gamma g}^{(1)} = 1 - c_g, \quad g \in G. \tag{5}$$

Define the symmetry operator by

$$\widehat{\gamma g} := \gamma \hat{g}. \tag{6}$$

Then the factor system of $G + \gamma G$ is

$$\hat{g}\hat{h} = z_{g,h}\widehat{gh}, \tag{7}$$

$$\hat{g}\widehat{\gamma h} = \hat{g}\gamma\hat{h} = (-1)^{c_g} z_{g,h}\widehat{\gamma g h}, \tag{8}$$

$$\widehat{\gamma g}\hat{h} = \gamma\hat{g}\hat{h} = z_{g,h}\widehat{\gamma g h}, \tag{9}$$

$$\widehat{\gamma g}\widehat{\gamma h} = \gamma\hat{g}\gamma\hat{h} = (-1)^{c_g} z_{g,h}\widehat{gh}. \tag{10}$$

If we define

$$\epsilon_g = \begin{cases} 0 & (g \in G), \\ 1 & (g \in \gamma G), \end{cases} \tag{11}$$

then the factor system $z_{g,h}^{(1)}$ of $G^{(1)}$ can be written compactly as

$$z_{g,h}^{(1)} = (-1)^{c_g \epsilon_h} z_{g,h}, \quad g, h \in G + \gamma G. \tag{12}$$

Next consider $n = 2$. Put $\gamma_1 = \sigma_x$, $\gamma_2 = \sigma_z$, $H = \sigma_y \otimes H'$, and $\hat{g} = \sigma_y^{c_g} \otimes \hat{g}'$. Then γ_1, γ_2 can be eliminated. The factor system obeyed by $\hat{g}'$ changes as follows:

$$\hat{g}\hat{h} = \sigma_y^{c_g} \otimes \hat{g}'\, \sigma_y^{c_h} \otimes \hat{h}' = (-1)^{\phi_g c_h} \sigma_y^{c_{gh}} \otimes \hat{g}'\hat{h}', \tag{13}$$

and hence

$$z^{(2)}_{g,h} = (-1)^{\phi_g c_h} z_{g,h}. \tag{14}$$

The value of c_g also changes. Since

$$\hat{g}H = \sigma_y^{c_g} \otimes \hat{g}'\, \sigma_y \otimes H' = (-1)^{\phi_g} \sigma_y^{1-c_g} \hat{g}'H', \tag{15}$$

$$H\hat{g} = \sigma_y \otimes H'\, \sigma_y^{c_g} \otimes \hat{g}' = \sigma_y^{1-c_g} \otimes H'\hat{g}', \tag{16}$$

we obtain

$$\hat{g}'H'\hat{g}'^{-1} = (-1)^{\phi_g + c_g} H'. \tag{17}$$

Thus

$$c^{(2)}_g = \phi_g + c_g \mod 2. \tag{18}$$

For $n \geq 3$ one repeats the above procedure. The result is

	$G^{(n)}$	$\phi^{(n)}_g$	$c^{(n)}_g$	$z^{(n)}_{g,h}$	γ
$n = 0$	G	ϕ_g	c_g	$z_{g,h}$	
$n = 1$	$G + \gamma G$	ϕ_g	$c_g + \epsilon_g$	$(-1)^{c_g \epsilon_h} z_{g,h}$	$\hat{g}\gamma = (-1)^{c_g} \gamma \hat{g}$
$n = 2$	G	ϕ_g	$c_g + \phi_g$	$(-1)^{\phi_g c_h} z_{g,h}$	
$n = 3$	$G + \gamma G$	ϕ_g	$c_g + \phi_g + \epsilon_g$	$(-1)^{(c_g+\phi_g)\epsilon_h + \phi_g c_h} z_{g,h}$	$\hat{g}\gamma = (-1)^{c_g + \phi_g} \gamma \hat{g}$
$n = 4$	G	ϕ_g	c_g	$(-1)^{\phi_g \phi_h} z_{g,h}$	
$n = 5$	$G + \gamma G$	ϕ_g	$c_g + \epsilon_g$	$(-1)^{c_g \epsilon_h + \phi_g \phi_h} z_{g,h}$	$\hat{g}\gamma = (-1)^{c_g} \gamma \hat{g}$
$n = 6$	G	ϕ_g	$c_g + \phi_g$	$(-1)^{\phi_g (c_h + \phi_h)} z_{g,h}$	
$n = 7$	$G + \gamma G$	ϕ_g	$c_g + \phi_g + \epsilon_g$	$(-1)^{(c_g+\phi_g)\epsilon_h + \phi_g (c_h + \phi_h)} z_{g,h}$	$\hat{g}\gamma = (-1)^{c_g + \phi_g} \gamma \hat{g}$.

Some comments:

- Let $G^{(n)}_0 = \ker \phi^{(n)} \cap \ker c^{(n)}$ be the subgroup that is unitary and commutes with the Hamiltonian. For even degrees, $G^{(n \in \text{even})}_0 = G_0$, and the factor system restricted to this subgroup is $z^{(n \in \text{even})}_{g,h} = z_{g,h}$. For odd degrees, $G^{(n \in \text{odd})}_0 = G_0 + \gamma \Gamma G_0$, and the factor system is $z^{(n \in \text{odd})}_{g,h} = (-1)^{c_g \epsilon_h} z_{g,h}$ on the corresponding subgroup. Therefore it is enough to construct the lattice of irreducible representations separately for the two cases: even and odd.
- Thus the irreducible representations of $G^{(n)}_0$ do not depend on the one-dimensional representation of the superconducting gap function.
- For even degree $(-n)$, the roles of TRS and PHS are exchanged.
- For odd degree $-n$, one would prefer not to add the chiral symmetry γ as a new symmetry, but it seems unavoidable.[1]

For odd degree $-n$, one should also define the corresponding, possibly half-translation, by

$$t_g = \begin{cases} t_g & g \in G, \\ t_{\gamma^{-1} g} & g \in \gamma G. \end{cases} \tag{19}$$

For reference, here is the summary of group elements in the normal and superconducting cases.

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[1] For the chiral index, see `http://www2.yukawa.kyoto-u.ac.jp/~ken.shiozaki/doc/chiral_index.pdf`.

- Normal state. With $G = G_0 + TG_0$,

$$G^{(n)} = \begin{cases} \{G_0, TG_0, 0, 0\} & (n = 0, 4), \\ \{G_0, TG_0, \gamma TG_0, \gamma G_0\} & (n = 1, 3), \\ \{G_0, 0, TG_0, 0\} & (n = 2, 6), \\ \{G_0, \gamma TG_0, TG_0, \gamma G_0\} & (n = 3, 7). \end{cases} \tag{20}$$

- Superconducting state.

$$G^{(n)} = \begin{cases} \{G_0, TG_0, CG_0, \Gamma G_0\} & (n = 0, 4), \\ \{G_0 + \Gamma\gamma G_0, T(G_0 + \Gamma\gamma G_0), C(G_0 + \Gamma\gamma G_0), \Gamma(G_0 + \Gamma\gamma G_0)\} & (n = 1, 3), \\ \{G_0, CG_0, TG_0, \Gamma G_0\} & (n = 2, 6), \\ \{G_0 + \Gamma\gamma G_0, C(G_0 + \Gamma\gamma G_0), T(G_0 + \Gamma\gamma G_0), \Gamma(G_0 + \Gamma\gamma G_0)\} & (n = 3, 7). \end{cases} \tag{21}$$