

# Contractivity of the Trace Norm

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**Definition 0.1** (Trace norm). For  $A \in \text{Mat}_n(\mathbb{C})$ , define

$$\|A\|_{\text{tr}} = \text{tr} |A|. \quad (1)$$

This is called the trace norm.

If  $\{\sigma_a\}_a$  are the singular values of  $A$ , then

$$\|A\|_{\text{tr}} = \sum_{a=1}^n \sigma_a. \quad (2)$$

Also note that, for arbitrary  $U, V \in U(n)$ ,

$$\|UAV^\dagger\|_{\text{tr}} = \|A\|_{\text{tr}}. \quad (3)$$

We first prove the following lemma.

**Lemma 0.2.** For  $A, B \in \text{Mat}_n(\mathbb{C})$ ,

$$|\text{tr}(AB)| \leq \|A\|_{\text{tr}} \|B\|. \quad (4)$$

Equality holds when  $A = U|A|$  is the polar decomposition and  $B = U^\dagger$ .

Write the polar decomposition as  $A = U|A|$ <sup>2</sup>. Consider the linear map

$$\phi_{|A|} : \text{Mat}_n(\mathbb{C}) \rightarrow \text{Mat}_n(\mathbb{C}), \quad \phi_{|A|}(B) := \text{tr}(|A|B) 1_n = \text{tr}(\sqrt{|A|}B\sqrt{|A|}) 1_n. \quad (5)$$

Since this map is positive, the Russo–Dye theorem implies that for any  $B \in \text{Mat}_n(\mathbb{C})$ ,

$$|\text{tr}(|A|B)| = \|\phi_{|A|}(B)\| \leq \|\phi_{|A|}\| \|B\| \leq \|\phi_{|A|}(1)\| \|B\| = \text{tr} |A| \|B\| = \|A\|_{\text{tr}} \|B\|. \quad (6)$$

Therefore

$$|\text{tr}(AB)| = |\text{tr}(|A|BU)| \leq \|A\|_{\text{tr}} \|BU\| = \|A\|_{\text{tr}} \|B\|. \quad (7)$$

Equality holds when  $BU = 1_n$ . □

This gives the variational expression for the trace norm:

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<sup>1</sup>The following proof was explained to me by Akihiro Hobashi. I am grateful to him.

<sup>2</sup>If  $A = U\Sigma V^\dagger$  is a singular value decomposition, then  $A = (UV^\dagger)V\Sigma V^\dagger$  is the polar decomposition.

**Corollary 0.3.**

$$\|A\|_{\text{tr}} = \sup_{B, \|B\|=1} |\text{tr}(AB)|. \quad (8)$$

We prove the following.

**Theorem 0.4** (Contractivity of the trace norm). *Let  $T : \text{Mat}_m(\mathbb{C}) \rightarrow \text{Mat}_n(\mathbb{C})$  be a positive linear map. Then, for  $A \in \text{Mat}_m(\mathbb{C})$ ,*

$$\|T(A)\|_{\text{tr}} \leq \|T^*(1)\| \|A\|_{\text{tr}}. \quad (9)$$

*In particular, if  $T$  is trace-preserving, then*

$$\|T(A)\|_{\text{tr}} \leq \|A\|_{\text{tr}}. \quad (10)$$

Indeed,

$$\|T(A)\|_{\text{tr}} = \sup_{B, \|B\|=1} |\text{tr}(T(A)B)| = \sup_{B, \|B\|=1} |\text{tr}(AT^*(B))|. \quad (11)$$

Here

$$|\text{tr}(AT^*(B))| \leq \|A\|_{\text{tr}} \|T^*(B)\| \leq \|A\|_{\text{tr}} \|T^*\| \|B\| = \|A\|_{\text{tr}} \|T^*(1)\| \|B\|, \quad (12)$$

and hence

$$\|T(A)\|_{\text{tr}} \leq \|A\|_{\text{tr}} \|T^*(1)\|. \quad (13)$$

□

# 1 Backup

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Write  $M_n = \text{Mat}_{n,n}(\mathbb{C})$ . For a linear map  $T : M_m \rightarrow M_n$ , define the dual map  $T^*$  by

$$\text{tr}[YT(X)] = \text{tr}[T^*(Y)X], \quad X \in M_m, \quad Y \in M_n. \quad (14)$$

If  $T$  is positive, this agrees with the adjoint operator with respect to the Hilbert–Schmidt inner product.

If  $T$  is positive, then  $T^*$  is also positive.

For  $X = \sum_i \lambda_i |i\rangle\langle i| \geq 0$ ,

$$\langle v|T^*(X)|v\rangle = \text{tr}[T^*(X)|v\rangle\langle v|] = \text{tr}[XT(|v\rangle\langle v|)] = \sum_i \lambda_i \langle i|T(|v\rangle\langle v|)|i\rangle. \quad (15)$$

Since  $\lambda_i \geq 0$  and  $T(|v\rangle\langle v|) \geq 0$ , all terms are nonnegative. □

A linear map  $T : M_m \rightarrow M_n$  is trace-preserving if and only if  $T^*$  is unital:

- $\text{tr} T(X) = \text{tr} X$  for  $X \in M_m$ ;
- $T^*(1) = 1$ .

If  $T$  is trace-preserving, then for arbitrary  $X \in M_m$ ,

$$\text{tr}[T^*(1)X] = \text{tr}[T(X)] = \text{tr}[X], \quad (16)$$

so  $T^*(1) = 1$ . Conversely, if  $T^*$  is unital, then

$$\text{tr}[T(X)] = \text{tr}[1T(X)] = \text{tr}[T^*(1)X] = \text{tr}[X]. \quad (17)$$

□

For a Hermitian matrix  $A$ , write its spectral decomposition as

$$A = \sum_i \lambda_i E_i, \quad (18)$$

and define

$$A_+ = \sum_{i, \lambda_i > 0} \lambda_i E_i, \quad A_- = \sum_{i, \lambda_i < 0} \lambda_i E_i, \quad P_{A>0} = \sum_{i, \lambda_i > 0} E_i. \quad (19)$$

We use the convention

$$A = A_+ - A_-, \quad |A| = \sqrt{A^\dagger A} = A_+ + A_-. \quad (20)$$

Then

For a Hermitian matrix  $A$ ,

$$\text{tr} A_+ = \max_{0 \leq S \leq 1} \text{tr} AS. \quad (21)$$

Equality holds for  $S = P_{A>0}$ .

<sup>3</sup>This note refers to <http://www.quest.lab.uec.ac.jp/ogawa/qmath2020/qmath20200729.pdf>.

Indeed,

$$\begin{aligned}
\operatorname{tr} AS &= \operatorname{tr} A_+ S - \operatorname{tr} A_- S \\
&= \operatorname{tr} [A_+] - \operatorname{tr} [A_+(1-S)] - \operatorname{tr} [A_- S] \\
&= \operatorname{tr} [A_+] - \operatorname{tr} [(1-S)^{1/2} A_+ (1-S)^{1/2}] - \operatorname{tr} [S^{1/2} A_- S^{1/2}].
\end{aligned} \tag{22}$$

For  $X \geq 0$ , one has  $Y^\dagger X Y \geq 0$ , since

$$\langle v | Y^\dagger X Y | v \rangle = \langle Y v | X | Y v \rangle \geq 0. \tag{23}$$

Therefore the last two trace terms are nonnegative, which proves the claim.  $\square$

Let  $T : M_m(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  be a trace-preserving positive linear map. For a Hermitian matrix  $A$ ,

$$\operatorname{tr} A_+ \geq \operatorname{tr} T(A)_+, \quad \operatorname{tr} A_- \geq \operatorname{tr} T(A)_-. \tag{24}$$

We have

$$\operatorname{tr} T(A)_+ = \operatorname{tr} [T(A) P_{T(A)>0}] = \operatorname{tr} [A T^*(P_{T(A)>0})]. \tag{25}$$

Since  $P_{T(A)>0} \leq 1$  and  $T^*$  is positive,

$$T^*(1 - P_{T(A)>0}) \geq 0. \tag{26}$$

As  $T^*$  is unital,

$$1 = T^*(1) \geq T^*(P_{T(A)>0}) \geq 0. \tag{27}$$

Thus (21) gives

$$\operatorname{tr} [A T^*(P_{T(A)>0})] \leq \max_{0 \leq S \leq 1} \operatorname{tr} [A S] = \operatorname{tr} A_+. \tag{28}$$

The proof for  $\operatorname{tr} A_-$  is identical.  $\square$

The desired statement is the following.

**Lemma 1.1** (Monotonicity of the trace norm). *Let  $T : M_m(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  be a trace-preserving positive linear map. If  $A \in M_n(\mathbb{C})$  is Hermitian, then*

$$\operatorname{tr} |A| \geq \operatorname{tr} |T(A)|. \tag{29}$$

Indeed,

$$\operatorname{tr} |A| = \operatorname{tr} (A_+ + A_-) = \operatorname{tr} (A_+) + \operatorname{tr} (A_-) \geq \operatorname{tr} T(A)_+ + \operatorname{tr} T(A)_- = \operatorname{tr} |T(A)|. \tag{30}$$

$\square$

It seems that the same statement holds for general  $A \in M_d(\mathbb{C})$ , but I could not prove it here.